

MIXTURE MODEL FOR DESIGNS IN HIGH DIMENSIONAL REGRESSION AND THE LASSO

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ABSTRACT. The LASSO is a recent technique for variable selection in the regression model

$$y = X\beta + \varepsilon,$$

where $X \in \mathbb{R}^{n \times p}$ and ε is a centered gaussian i.i.d. noise vector $\mathcal{N}(0, \sigma^2 I)$. The LASSO has been proved to perform exact support recovery for regression vectors when the design matrix satisfies certain algebraic conditions and β is sufficiently sparse. Estimation of the vector $X\beta$ has also extensively been studied for the purpose of prediction under the same algebraic conditions on X and under sufficient sparsity of β . Among many other, the coherence is an index which can be used to study these nice properties of the LASSO. More precisely, a small coherence implies that most sparse vectors, with less nonzero components than the order $n/\log(p)$, can be recovered with high probability if its nonzero components are larger than the order $\sigma\sqrt{\log(p)}$. However, many matrices occurring in practice do not have a small coherence and thus, most results which have appeared in the litterature cannot be applied. The goal of this paper is to study a model for which precise results can be obtained. In the proposed model, the columns of the design matrix are drawn from a Gaussian mixture model and the coherence condition is imposed on the much smaller matrix whose columns are the mixture's centers, instead of on X itself. Our main theorem states that $X\beta$ is as well estimated as in the case of small coherence up to a correction parametrized by the maximal variance in the mixture model.

1. INTRODUCTION

The goal of the present paper is the study of the high dimensional regression problem $y = X\beta + z$, where $X \in \mathbb{R}^{n \times p}$, with $p \gg n$ and $z \sim \mathcal{N}(0, \sigma^2 I_n)$. For simplicity, we will assume throughout this paper that the columns of X have unit l_2 -norm. This problem has been the subject of a great research activity. This high dimensional setting, where more variables are involved than observations, occurs in many different applications such as image processing and denoising, gene expression analysis, and, after slight modifications, time series (filtering) [17], [20], machine learning and especially graphical models [19] and more recently, biochemistry [1]. One of the most popular approaches is the Least Angle Shrinkage and Selection Operator (LASSO) introduced in [23] for the purpose of variable selection. The LASSO estimator is given as a solution, for $\lambda > 0$, of

$$(1.1) \quad \hat{\beta} = \underset{b \in \mathbb{R}^p}{\operatorname{Argmin}} \frac{1}{2} \|y - Xb\|_2^2 + \lambda \|b\|_1.$$

Conditions for uniqueness of the minimizer in this last expression are discussed in [16], [21] and [15]. Several other estimators have also been proposed, such as the Dantzig Selector [11] [3] or Message Passing Algorithms [14]. In the sequel, we will focus on the LASSO due to its wide use in various applications.

One of the most surprising and important discoveries from these recent extensive efforts is that, under appropriate assumptions on the design matrix X , and for most regression vectors β , the support of β can be recovered exactly when its size is of the order $n/\log(p)$; see [3], [5], [9], [28] for instance. Moreover, under similar assumptions, the prediction error can be controlled adaptively as a function of the sparsity of β and the noise variance; see for instance [9]. Similar rates can be achieved by other method, involving for instance penalization, but the main advantage of the LASSO over most competitors is that a solution can be obtained in polynomial time, following the definition of complexity theory. A very efficient algorithm is, e.g., [2]. Many implementations are available on the web.

The two main assumptions for achieving these remarkable results are unavoidably imposed on the design matrix X and on the regression vector.

- The regression vector β is assumed to be s -sparse, with support denoted by T , meaning that no more than s of its components are non zero. This can be relaxed to β assumed only compressible, that is approximable by a sparse vector.
- The design matrix is assumed to satisfy one of many proposed algebraic conditions in the litterature, implying that all singular values of X_S are close to one for any or most given $S \subset \{1, \dots, n\}$ with $|S| = s'$ for some appropriate choice of s' (often equal to s or $2s$).

Concerning the second point, two main assumptions have been proposed in the litterature. The first is the Restricted Isometry Property [10] [8], which requires that

$$(1.2) \quad (1 - \delta)\|\beta_S\|_2^2 \leq \|X_S\beta_S\|_2^2 \leq (1 + \delta)\|\beta_S\|_2^2,$$

for $S \subset \{1, \dots, n\}$ with $|S| = s'$ and all $\beta \in \mathbb{R}^p$. This property is satisfied by high probability for most random matrices with i.i.d. entrees with variance $1/n$ such as Gaussian or Rademacher variables and for $s' \leq C_{rip} n/\log(p)$, where the constant C_{rip} depends on the distribution of the individual entrees. Notice that the $1/n$ assumption on the variance and standard concentration bounds imply that the resulting random matrix has almost normalized columns and the normalized avatar will satisfy RIP with unessential modifications of the constants. The RIP has been extensively used in signal processing after the emergence of the so-called Compressed Sensing paradigm [7].

The second assumption which is often considered is the Incoherence Condition, which requires that

$$\mu(X) = \max_{j \neq j'=1}^p |\langle X_j, X_{j'} \rangle|$$

is small, e.g. $\mu(X) \leq C_\mu/\log(p)$ as in [9], which is guaranteed for random matrices with i.i.d. gaussian entrees with variance $1/n$ in the range $n \geq C_{ic} \log(p)^3$.

The main advantage of the Incoherence Condition over the Restricted Isometry Property is that it can be checked quickly (in $p(p-1)/2$ operations), whereas no-one knows how to check the RIP without enumerating all possible supports $S \subset \{1, \dots, n\}$ with cardinal s' . Such an enumeration would of course take an exponential amount of time to establish. The main relationship between IC and RIP is that it can be proved that under IC, (1.2) holds, not for all, but for most supports $S \subset \{1, \dots, n\}$ with cardinal s' , where $s' \leq C_s p/(\|X\| \log(p))$, for some constant C_s controlling the proportion of such supports.

The objective of the present paper is to extend the analysis based on the Incoherence Condition to more general situations where X may have a lot of very colinear columns. The main idea is to assume that the columns are drawn from a mixture model of K clusters, and that the set of cluster's centers form a matrix which satisfies the Incoherence Condition.

2. MAIN RESULTS

2.1. The mixture model. In order to relax the Incoherence Condition, one needs a model for the design matrix X allowing for a certain amount of correlations between columns while keeping some of the algebraic structure in the same spirit as (1.2) for at least most supports indexing a subset of really pertinent covariates. In what follows, we study such a model, where the columns can be considered as belonging to a family of clusters and the cluster's centers or (an empirical surrogate) is defined to be the pertinent variable. This model is of great interest when many columns are very colinear. In practice, one often observes that the columns of X can be grouped into different clusters such that the dot product of X_j and $X_{j'}$ $j \neq j'$ is close to one if they belong to the same cluster, and very close to zero otherwise. Notice that applying the LASSO for such designs will eventually result into grossly incorrect variable selection. On the other hand confusing a variable for another very correlated variable might not be a real issue as far as prediction is concerned if the clusters are well separated.

2.1.1. Detailed presentation. Let K be the number of clusters in the covariates. Consider a matrix \mathfrak{C} in $\mathbb{R}^{n \times K}$, with small coherence. The columns of the matrix \mathfrak{C} will be the "centers" of each cluster, $k = 1, \dots, K$.

The design matrix will be assumed to derive from a matrix X_o whose columns are drawn from the following procedure. Let \mathcal{K} be randomly drawn among all index subsets of $\{1, \dots, K\}$ with cardinal

s^* with uniform distribution. We then assume that, conditionally on \mathcal{K} each column of X_o is drawn from a mixture Φ of K n -dimensional Gaussian distributions, i.e.

$$\Phi(x) = \sum_{k \in \mathcal{K}} \pi_k \phi_k(x),$$

where

$$\phi_k(x) = \frac{1}{(2\pi\mathfrak{s}^2)^{\frac{n}{2}}} \exp\left(-\frac{\|x - \mathfrak{C}_k\|_2^2}{2\mathfrak{s}^2}\right),$$

and $\pi_k \geq 0$, $k \in \mathcal{K}$ and $\sum_{k \in \mathcal{K}} \pi_k = 1$. We will denote by n_k the random number of columns in X_o that were drawn from $\mathcal{N}(\mathfrak{C}_k, \mathfrak{s}^2 \text{Id})$, $k = 1, \dots, K_o$. Thus, $\sum_{k \in \mathcal{K}} n_k = p$.

Finally, the matrix X is obtained by column-wise normalization of X_o , i.e. $X_j = X_{o,j} / \|X_{o,j}\|_2$.

Notice that the model could easily be modified in order to more general distributions for \mathcal{K} than the uniform distribution on subsets of $\{1, \dots, K_o\}$ with cardinal s^* .

2.1.2. More notations. For each $j \in \{1, \dots, p\}$, denote by k_j the index of the Gaussian component from which columns j was drawn, and let J_k denote the set of indices of columns drawn from the k^{th} Gaussian component. For any index set $S \in \{1, \dots, p\}$, let \mathcal{K}_S denote the list (with possible repetitions)

$$\mathcal{K}_S = \{k_j \mid j \in S\}.$$

The deviation of columns $X_{o,j}$ from center \mathfrak{C}_{k_j} will be denoted by

$$\varepsilon_j = X_{o,j} - \mathfrak{C}_{k_j} \sim \mathcal{N}(\mathfrak{C}_{k_j}, \mathfrak{s}^2).$$

and the matrix E is defined as

$$E = (\varepsilon_{i,j})_{i \in \{1, \dots, n\}, j \in \{1, \dots, p\}}.$$

2.2. A simple proxy for β . For each $k \in \{1, \dots, K\}$, let j_k^* be the best approximation of the center \mathfrak{C}_k from the set of columns X_j , $j \in J_k$, i.e.

$$j_k^* = \underset{j \in J_k}{\text{Argmin}} \|X_j - \mathfrak{C}_k\|_2.$$

Moreover, set

$$T^* = \{j_k^* \mid k \in \mathcal{K}\}.$$

Of course, we have $s^* = |T^*|$.

The vector β^* is defined by

$$(2.3) \quad \mathfrak{C}_{\mathcal{K}_{T^*}} \beta_{T^*}^* = \mathfrak{C}_{\mathcal{K}_T} \beta_T.$$

A simple expression of β^* can be obtained by taking

$$(2.4) \quad \beta_{j^*}^* = \sum_{j \in J_{k_{j^*}} \cap T} \beta_j$$

for all $j^* \in T^*$. Moreover, this expression is unique whenever X_{T^*} has rank equal to s^* . In Section 3.1, we will show that X_{T^*} is indeed non-singular with high probability under appropriate assumptions on T .

2.3. Main result.

2.3.1. *Further notations.* In the sequel r will denote a constant in $(1, 1/4)$. The constants ϑ_* et ν will be specified in Assumptions 2.3 below. The constants C_μ , C_{spar} et C_{col} will be used in the Assumptions below:

$$C_\mu = r/(1 + \alpha),$$

$$C_{\text{spar}} = r^2/((1 + \alpha)e^2),$$

$$C_{\text{col}} = \frac{1}{2} \left(\frac{\sqrt{2}}{\sqrt{(1-r)(1+\alpha)}} - (1+r) \right).$$

Let C_χ denote a positive constant such that

$$\mathbb{P} \left(\frac{\|G\|_2^2}{\mathfrak{s}^2} \leq u^2 \right) \leq C_\chi \left(\frac{u^2}{n} \right)^n$$

where G is a n -dimensional centered and unit-variance i.i.d. gaussian vector. Let us further define

$$(2.5) \quad r_{\max} = 1 + \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right),$$

$$(2.6) \quad \mu_{\max} = \frac{1}{2} \mathfrak{s} \left(\sqrt{n} + \sqrt{s} + \sqrt{2\alpha \log(p)} \right),$$

$$(2.7) \quad \sigma_{\max}^2 = \frac{1}{2} \sqrt{s} \mathfrak{s}^2,$$

$$r_{\max}^* = \frac{1}{1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}},$$

$$(2.8) \quad K_{n,s^*}^2 = \alpha n \log(p) \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}},$$

$$\mu_{\max}^* = \mathfrak{s} K_{n,s^*},$$

$$(2.9) \quad \sigma_{\max}^{*2} = \frac{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}{1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}} \sqrt{s^*} \mathfrak{s}^2,$$

$$C_f = \int_0^3 \sqrt{\log \left(\frac{3}{\varepsilon'} \right)} d\varepsilon'.$$

and

$$C_f^* = \int_0^3 \sqrt{\log \left(\frac{3}{\varepsilon'} \right)} d\varepsilon'.$$

2.3.2. *Assumptions.* We will make the following assumptions.

Assumptions 2.1.

$$p \geq \max \left\{ K_o, e^{e^2 - \log(\alpha)} \right\}.$$

and

$$\log(p) \geq \max \left\{ \frac{0.2 \cdot r (1 + 1.1 \cdot r + 0.11 \cdot r^2)}{0.1 \cdot (1.1 \cdot r + 0.11 \cdot r^2)}, \frac{1.1 \cdot r (1.1 + 0.11 \cdot r)}{\alpha} \right\}.$$

Assumptions 2.2. Assume that \mathfrak{C} has coherence $\mu(\mathfrak{C})$ satisfying

$$(2.10) \quad \mu(\mathfrak{C}) \leq \frac{C_\mu}{\log(p)}.$$

Assumptions 2.3. There exists a positive real constant ϑ^* and a positive integer ν such that

$$\min_{j^* \in T^*} |\mathcal{J}_{k_{j^*}}| \geq \vartheta_* \log(p)^\nu.$$

Assumptions 2.4.

$$s^* \leq \frac{K_o}{\log p} \frac{C_{\text{spar}}}{\|\mathfrak{C}\|^2}.$$

Assumptions 2.5.

$$(2.11) \quad n \geq \frac{\alpha + 1}{c} \log(p).$$

Remark 2.1. Assumption 2.5 is to be interpreted with care since the order of magnitude of n is primarily governed by Assumption 2.2 on the coherence of \mathfrak{C} . For instance, if \mathfrak{C} comes from a Gaussian i.i.d. random matrix, the coherence will be of the order $\sqrt{\log(K_o)/n}$ as discussed in [9, Section 1.1] and n should be at least of the order $\log(p)^2 \log(K_o)$. Notice that this is still less than if X itself had to satisfy the coherence bound, which would imply that n be of the order $\log(p)^3$.

Assumptions 2.6.

$$C_{\text{col}} \geq e^2(\alpha + 1) \max\{\sqrt{C_{\text{spar}}}, C_\mu\}.$$

and

$$(C_{\text{col}} + (1 + 1.1 \cdot r) C_{\mathfrak{s},n,p}) \leq \frac{1}{2} \sqrt{\frac{\log(p) (1 - r^*)^2}{(\alpha \log(p) - \log(2)) 2}}.$$

Assumptions 2.7. $\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \leq 1/2$

$$(2.12) \quad \mathfrak{s} \leq C_{\mathfrak{s},n,p} \frac{1}{\sqrt{\log(p)} \left(\sqrt{n} + \sqrt{\frac{\alpha+1}{c} \log(p)} \right)}.$$

for any $C_{\mathfrak{s},n,p}$ such that and

$$C_{\mathfrak{s},n,p} \leq \min \left\{ 0.1 \cdot \frac{r}{\sqrt{\alpha \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}; \frac{1}{2} \sqrt{\log(p)} \right\}.$$

Assumptions 2.8.

$$\|\beta_T\|_2^2 \geq \frac{2 \alpha \log(p) n \sigma_{\max}^2}{\frac{4 \alpha^2}{9} \mu_{\max}^2 \log^2(p) - 12 C_f \mu_{\max} r_{\max} \mathfrak{s} \sqrt{n}}.$$

Assumptions 2.9.

$$\|\beta_{T^*}^*\|_2^2 \geq \frac{2 \alpha \log(p) n \sigma_{\max}^{*2}}{\frac{4 \alpha^2}{9} \mu_{\max}^{*2} \log^2(p) - 24 C_f^* \mu_{\max}^* r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}.$$

Remark 2.2. Notice that, using (2.4), the following relationship holds between $\|\beta_{T^*}^*\|_2^2$ and $\|\beta_T\|_2^2$ if all the coefficients β_j , $j \in \mathcal{J}_k$ have the same sign for all $k \in \mathcal{K}$:

$$\|\beta_{T^*}^*\|_2^2 \geq \|\beta_T\|_2^2.$$

In this case, one can replace $\|\beta_{T^*}^*\|_2^2$ by $\|\beta_T\|_2^2$ in Assumption (2.9) and merge Assumption (2.8) and Assumption (2.9) by taking the maximum of their respective right hand side and obtain a simpler assumption.

Assumptions 2.10. The support of $\beta_{T^*}^*$ is random and uniformly distributed among subsets of $\{1, \dots, p\}$ with cardinal s^* . The sign of $\beta_{T^*}^*$ is random with uniform distribution on $\{-1, 1\}^{s^*}$.

Remark 2.3. This last assumption is a transposition to the proxy β^* of the conditions on β in [9].

2.3.3. *Main theorem.* The main result of this paper is the following theorem.

Theorem 2.4. Set $\lambda = 2\sigma\sqrt{2\alpha \log(p)}$. Assume that X is drawn from the Gaussian mixture model of Section 2.1 with \mathcal{K} drawn uniformly at random among all possible index subsets of $\{1, \dots, K\}$ with cardinal s^* . Let Assumptions 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9 hold. Then, we have

$$\frac{1}{2}\|Xh\|_2^2 \leq s^* \frac{3}{2} r_* \lambda \left(\frac{3}{2} \lambda + \sqrt{1+r_*} \delta \|\mathfrak{C}_T \beta_T\|_2 \right) + \frac{1}{2} \delta^2 \|X\beta\|_2^2$$

with $r_* = 1.1 \cdot r$ ($1.1 + 0.11 \cdot r$) and for any δ satisfying

$$\begin{aligned} \delta \geq & 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \left(1 + 8\sqrt{2} \sqrt{\alpha \log(p) + \log(2n+2)} \sqrt{s^* \rho \mathfrak{C}} \right) \\ & + \left(12 C_f \mathfrak{s} \sqrt{n} r_{\max} + \alpha \log(p) \mu_{\max} \right) \sqrt{s^* \rho \mathfrak{C}} \\ & + 4\mathfrak{s} \sqrt{n \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \left(1 + 2\sqrt{2} \sqrt{\rho \mathfrak{C}} \sqrt{\alpha \log(p) + \log(2n+2)} \right)} \\ (2.13) \quad & + \left(24 r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} C_f^* + \mu_{\max}^* \alpha \log(p)} \right) \sqrt{\rho \mathfrak{C}}, \end{aligned}$$

3. PROOF OF THEOREM 2.4

Some parts of the proof closely follow the key arguments in the proof of [9, Theorem 1.2]. Their adaptation to the present setting is however sometimes nontrivial. We present all the details for the sake of completeness.

3.1. Preliminaries: Candès and Plan's conditions. The following proposition will be much used in the arguments.

Proposition 3.1. We have the following properties:

(I)

$$(3.14) \quad \mathbb{P} \left(\|\mathfrak{C}_\mathcal{K}^t \mathfrak{C}_\mathcal{K} - \text{Id}_s\| \geq \frac{1}{2} \right) \leq \frac{216}{p^\alpha}.$$

(II)

$$(3.15) \quad \mathbb{P} \left(\|X_{T^*}^t X_{T^*} - I\| \geq 1.1 \cdot r(1.1 + 0.11 \cdot r) \right) \leq \frac{219}{p^\alpha}.$$

(III)

$$(3.16) \quad \mathbb{P} \left(\|X^t z\|_\infty \geq \sigma \sqrt{2\alpha \log(p)} \right) \leq \frac{1}{p^\alpha}.$$

(IV)

$$\begin{aligned}
& \|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t z\|_\infty + \lambda \|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} \text{sgn}(\beta_{T^*}^*)\|_\infty \\
(3.17) \quad & \leq \sigma \sqrt{1 + 1.1 \cdot r(1.1 + 0.11 \cdot r)} + \frac{1}{2} \lambda
\end{aligned}$$

Proof. See Appendix C. □

3.2. Controlling $\|X\beta - X\beta^*\|_2$ by $\|X\beta\|$.

Proposition 3.2. *One has*

$$\mathbb{P}(\|X\beta - X\beta^*\|_2 \geq \delta \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2) \leq \frac{1}{p^\alpha}$$

Proof. The proof is divided into four steps, for the sake of clarity.

Step 1. Let

$$\tilde{E}_T = X_T - \mathfrak{C}_{\mathcal{K}_T}.$$

where, since \mathcal{K}_T is supposed to be a list with possible repetitions, the matrix $\mathfrak{C}_{\mathcal{K}_T}$ has correspondingly possible column repetitions, and

$$\tilde{E}_T^* = X_{T^*} - \mathfrak{C}_{\mathcal{K}_{T^*}}.$$

Thus, using (2.3),

$$\|X\beta - X\beta^*\|_2 = \|\tilde{E}_T \beta_T - \tilde{E}_T^* \beta_{T^*}^*\|_2,$$

which, by the triangular inequality, gives

$$\|X\beta - X\beta^*\|_2 \leq \|\tilde{E}_T \beta_T\|_2 + \|\tilde{E}_T^* \beta_{T^*}^*\|_2.$$

Step 2: Control of $\|\tilde{E}_T \beta_T\|_2$. The column $j \in T$ of the matrix \tilde{E}_T has the expression

$$\tilde{E}_j = \frac{\mathfrak{C}_{k_j} + E_j}{\|\mathfrak{C}_{k_j} + E_j\|_2} - \mathfrak{C}_{k_j}.$$

We may decompose the quantity $\|\tilde{E}_T \beta_T\|_2$ as

$$\|\tilde{E}_T \beta_T\|_2 = \|A\|_2 + \|B\|_2,$$

where

$$A = \sum_{j \in T} \left(\frac{1}{\|\mathfrak{C}_{k_j} + E_j\|_2} - 1 \right) \mathfrak{C}_{k_j} \beta_j$$

and

$$B = \sum_{j \in T} \frac{1}{\|\mathfrak{C}_{k_j} + E_j\|_2} E_j \beta_j.$$

We have the following bound for A .

Lemma 3.3.

$$\begin{aligned}
(3.18) \quad & \mathbb{P} \left(\|A\|_2 \geq 4s \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \left(1 + 8\sqrt{2} \sqrt{\alpha \log(p) + \log(2n+2)} \sqrt{s^* \rho_{\mathfrak{C}}} \right) \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \right) \\
& \leq \frac{C+1}{p^\alpha}.
\end{aligned}$$

Proof. See Appendix A.1. □

Turning to B , we have the following result.

Lemma 3.4. *We have*

$$\mathbb{P} \left(\|B\|_2 \geq \left(12 C_f s \sqrt{n} r_{\max} + \alpha \log(p) \mu_{\max} \right) \sqrt{s^* \rho_{\mathfrak{C}}} \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \right) \leq \frac{2}{p^\alpha}.$$

Proof. See Appendix A.2. □

Step 3: Control of $\|\tilde{E}_{T^*}^* \beta_{T^*}^*\|_2$. The column $j^* \in T^*$ of the matrix $\tilde{E}_{T^*}^*$ has the expression

$$\tilde{E}_{j^*}^* = \frac{\mathfrak{e}_{k_{j^*}} + E_{j^*}}{\|\mathfrak{e}_{k_{j^*}} + E_{j^*}\|_2} - \mathfrak{e}_{k_{j^*}}.$$

We will procede as in Step 2. Define

$$W_{j^*}^* = \frac{1}{\|\mathfrak{e}_{k_{j^*}} + E_{j^*}\|_2} - 1.$$

Notice that $\tilde{E}_{j^*}^*$ can be written

$$\tilde{E}_{T^*}^* \beta_{T^*}^* = A^* + B^*$$

with

$$A^* = \sum_{j^* \in T^*} W_{j^*}^* \beta_{j^*}^* A_{j^*}^*,$$

where

$$A_{j^*}^* = \begin{bmatrix} 0 & \mathfrak{e}_{k_{j^*}}^t \\ \mathfrak{e}_{k_{j^*}}^t & 0 \end{bmatrix}.$$

and

$$B^* = \sum_{j^* \in T^*} \beta_{j^*}^* B_{j^*}^*, \text{ where } B_{j^*}^* = \frac{E_{j^*}}{\|\mathfrak{e}_{k_{j^*}} + E_{j^*}\|_2}.$$

We begin with the study of A^* .

Lemma 3.5. *We have*

$$\begin{aligned} \mathbb{P}\left(\|A^*\|_2 \geq 4\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} \right. \\ \left. \left(1 + 2\sqrt{2} \sqrt{\rho\mathfrak{e}} \sqrt{\alpha \log(p) + \log(2n+2)}\right) \|\mathfrak{e}_{\mathcal{K}_T} \beta_T\|_2\right) \\ \leq \frac{2}{p^\alpha}. \end{aligned}$$

Proof. See Appendix A.3. □

Turning to B^* , we have the following result.

Lemma 3.6. *We have*

$$\begin{aligned} \mathbb{P}\left(\|B^*\| \geq \left(24 r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^* \right. \right. \\ \left. \left. + \mu_{\max}^* \alpha \log(p)\right) \sqrt{\rho\mathfrak{e}} \|\mathfrak{e}_T \beta_T\|_2\right) \leq \frac{2}{p^\alpha}. \end{aligned}$$

Proof. See Appendix A.4. □

Step 4: Conclusion. Combining Lemmæ 3.3, 3.4, 3.5 and 3.6, we obtain that for any δ such that (2.13) we have

$$\mathbb{P}(\|X\beta - X\beta^*\|_2 \geq \delta \|\mathfrak{e}_T \beta_T\|_2) \leq \frac{1}{p^\alpha}.$$

□

3.3. The prediction bound. By definition, the LASSO estimator satisfies

$$(3.19) \quad \frac{1}{2}\|y - X\hat{\beta}\|_2^2 + \lambda\|\hat{\beta}\|_1 \leq \frac{1}{2}\|y - X\beta^*\|_2^2 + \lambda\|\beta^*\|_1.$$

One may introduce $X\beta$ in this expression and obtain

$$\frac{1}{2}\|y - X\beta + X(\beta - \hat{\beta})\|_2^2 + \lambda\|\hat{\beta}\|_1 \leq \frac{1}{2}\|y - X\beta + X(\beta - \beta^*)\|_2^2 + \lambda\|\beta^*\|_1,$$

from which we deduce

$$(3.20) \quad \begin{aligned} \frac{1}{2}\|X(\beta - \hat{\beta})\|_2^2 &\leq \langle y - X\beta, X(\hat{\beta} - \beta^*) \rangle \\ &\quad - \lambda \left(\|\hat{\beta}\|_1 - \|\beta^*\|_1 \right) + \frac{1}{2}\|X(\beta - \beta^*)\|_2^2. \end{aligned}$$

Set $h^* := \hat{\beta} - \beta^*$. Using sparsity of β^* , we obtain that $h^*_{T^{*c}} = \hat{\beta}_{T^{*c}} - \beta^*_{T^{*c}} = \hat{\beta}_{T^{*c}}$. Thus, we have

$$\begin{aligned} \|\hat{\beta}\|_1 - \|\beta^*\|_1 &= \|\beta^* + h^*\|_1 - \|\beta^*\|_1 \\ &= \|\beta^*_{T^*} + h^*_{T^*}\|_1 + \|\beta^*_{T^{*c}} + h^*_{T^{*c}}\|_1 - \|\beta^*_{T^*}\|_1 \\ &= \|\beta^*_{T^*} + h^*_{T^*}\|_1 - \|\beta^*_{T^*}\|_1 + \|h^*_{T^{*c}}\|_1. \end{aligned}$$

Since, for any b with no zero component, the gradient of $\|\cdot\|_1$ at b is $\text{sgn}(b)$, the subgradient inequality gives

$$\|\beta^*_{T^*} + h^*_{T^*}\|_1 \geq \|\beta^*_{T^*}\|_1 + \langle \text{sgn}(\beta^*_{T^*}), h^*_{T^*} \rangle$$

and combining this latter inequality with (3.20), we obtain

$$(3.21) \quad \begin{aligned} \frac{1}{2}\|X(\beta - \hat{\beta})\|_2^2 &\leq \langle y - X\beta, Xh^* \rangle - \lambda \langle \text{sgn}(\beta^*_{T^*}), h^*_{T^*} \rangle \\ &\quad - \lambda \|h^*_{T^{*c}}\|_1 + \frac{1}{2}\|X(\beta - \beta^*)\|_2^2. \end{aligned}$$

Set $r := \beta^* - \beta$ and $h := \hat{\beta} - \beta$. Using these notations, equation (3.21) may be written

$$(3.22) \quad \begin{aligned} \frac{1}{2}\|Xh\|_2^2 &\leq \langle z, Xh^* \rangle - \lambda \langle \text{sgn}(\beta^*_{T^*}), h^*_{T^*} \rangle \\ &\quad - \lambda \|h^*_{T^{*c}}\|_1 + \frac{1}{2}\|Xr\|_2^2. \end{aligned}$$

Using the fact that

$$\langle X^t z, h^* \rangle = \langle X_{T^*}^t z, h^*_{T^*} \rangle + \langle X_{T^{*c}}^t z, h^*_{T^{*c}} \rangle$$

and the following majorization based on (3.16)

$$\begin{aligned} \langle X_{T^{*c}}^t z, h^*_{T^{*c}} \rangle &\leq \|h^*_{T^{*c}}\|_1 \|X_{T^{*c}}^t z\|_\infty \\ &\leq \frac{1}{2} \lambda \|h^*_{T^{*c}}\|_1, \end{aligned}$$

we obtain that

$$\frac{1}{2}\|Xh\|_2^2 \leq \langle v, h^*_{T^*} \rangle - \left(1 - \frac{1}{2}\right) \lambda \|h^*_{T^{*c}}\|_1 + \frac{1}{2}\|Xr\|_2^2,$$

where $v := X_{T^*}^t z - \lambda \text{sgn}(\beta^*_{T^*})$.

Now, observe that

$$\begin{aligned} \langle v, h^*_{T^*} \rangle &= \langle v, (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t X_{T^*} h^*_{T^*} \rangle \\ &= \langle (X_{T^*}^t X_{T^*})^{-1} v, X_{T^*}^t X_{T^*} h^*_{T^*} \rangle \\ &= \underbrace{\langle (X_{T^*}^t X_{T^*})^{-1} v, X_{T^*}^t X h^* \rangle}_{A_1} - \underbrace{\langle (X_{T^*}^t X_{T^*})^{-1} v, X_{T^*}^t X_{T^{*c}} h^*_{T^{*c}} \rangle}_{A_2}. \end{aligned}$$

Let us begin by studying A_2 . We have that

$$\begin{aligned}
A_2 &\geq -\|X_{T^*c}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} v\|_\infty \|h_{T^*c}\|_1 \\
&\geq -\|X_{T^*c}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t z\|_\infty \|h_{T^*c}\|_1 \\
&\quad -\lambda \|X_{T^*c}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} \text{sgn}(\beta_{T^*}^*)\|_\infty \|h_{T^*c}^*\|_1 \\
&\geq -\left(\sigma \sqrt{1 + 1.1 \cdot r (1.1 + 0.11 \cdot r)} + \frac{1}{2} \lambda\right) \|h_{T^*c}^*\|_1
\end{aligned}$$

by (3.17). Thus

$$\langle v, h_{T^*}^* \rangle \leq A_1 + \left(\sigma \sqrt{1 + 1.1 \cdot r (1.1 + 0.11 \cdot r)} + \frac{1}{2} \lambda\right) \|h_{T^*c}\|_1$$

and since, by (2.10),

$$\left(\sigma \sqrt{1 + 1.1 \cdot r (1.1 + 0.11 \cdot r)} + \frac{1}{2} \lambda\right) \leq \lambda,$$

we deduce that

$$\frac{1}{2} \|Xh\|_2^2 \leq A_1 + \frac{1}{2} \|Xr\|_2^2,$$

Let us now bound A_1 from above. We have that

$$A_1 \leq \underbrace{\|X_{T^*}^t Xh^*\|_\infty}_{B_1} \underbrace{\|(X_{T^*}^t X_{T^*})^{-1} v\|_1}_{B_2}$$

Firstly,

$$\begin{aligned}
B_1 &\leq \|X_{T^*}^t (X\beta^* - y)\|_\infty + \|X_{T^*}^t (X\hat{\beta} - y)\|_\infty \\
&\leq \|X_{T^*}^t (Xr - z)\|_\infty + \|X_{T^*}^t (y - X\hat{\beta})\|_\infty \\
&\leq \frac{1}{2} \lambda + \|X_{T^*}^t Xr\|_\infty + \lambda
\end{aligned}$$

where we used (3.16), and the optimality condition for the LASSO estimator. Secondly,

$$\begin{aligned}
B_2 &\leq \sqrt{s^*} \|(X_{T^*}^t X_{T^*})^{-1} v\|_2 \\
&\leq \sqrt{s^*} \|(X_{T^*}^t X_{T^*})^{-1} \|v\|_2 \\
&\leq s^* \|(X_{T^*}^t X_{T^*})^{-1} \|v\|_\infty.
\end{aligned}$$

Moreover, (3.15) gives $\|(X_{T^*}^t X_{T^*})^{-1}\| \leq 1.1 \cdot r (1.1 + 0.11 \cdot r)$ and

$$\|v\|_\infty \leq \|X_{T^*}^t z\|_\infty + \lambda \leq \frac{3}{2} \lambda$$

Thus, we obtain that

$$A_1 \leq s^* 1.1 \cdot r (1.1 + 0.11 \cdot r) \frac{3}{2} \lambda \left(\frac{3}{2} \lambda + \|X_{T^*}^t Xr\|_\infty\right)$$

and thus,

$$\frac{1}{2} \|Xh\|_2^2 \leq s^* 1.1 \cdot r (1.1 + 0.11 \cdot r) \frac{3}{2} \lambda \left(\frac{3}{2} \lambda + \|X_{T^*}^t Xr\|_\infty\right) + \frac{1}{2} \|Xr\|_2^2.$$

Since $\|X_{T^*}^t Xr\|_\infty \leq \|X_{T^*}^t Xr\|_2$ and since $\|X_{T^*}^t Xr\|_2 \leq \sqrt{1 + 1.1 \cdot r (1.1 + 0.11 \cdot r)} \|Xr\|_2$, we obtain

$$\frac{1}{2} \|Xh\|_2^2 \leq s^* 1.1 \cdot r (1.1 + 0.11 \cdot r) \frac{3}{2} \lambda \left(\frac{3}{2} \lambda + \sqrt{1 + 1.1 \cdot r (1.1 + 0.11 \cdot r)} \|Xr\|_2\right) + \frac{1}{2} \|Xr\|_2^2.$$

Moreover, Proposition 3.2 yields

$$\frac{1}{2} \|Xh\|_2^2 \leq s^* 1.1 \cdot r (1.1 + 0.11 \cdot r) \frac{3}{2} \lambda \left(\frac{3}{2} \lambda + \sqrt{1 + 1.1 \cdot r (1.1 + 0.11 \cdot r)} \delta \|\mathfrak{C}_T \beta_T\|_2\right) + \frac{1}{2} \delta^2 \|X\beta\|_2^2$$

which completes the proof.

APPENDIX A. TECHNICAL LEMMÆ

A.1. Proof of Lemma 3.3. We have that

$$\|\mathfrak{C}_{k_j}\|_2 - \|E_j\|_2 \leq \|\mathfrak{C}_{k_j} + E_j\|_2 \leq \|\mathfrak{C}_{k_j}\|_2 + \|E_j\|_2.$$

Moreover, since $\|E_j\|_2^2/\mathfrak{s}^2$ follows the χ_n^2 -distribution, the scalar Chernov bound gives

$$(A.23) \quad \mathbb{P}\left(\left|\frac{\|E_j\|_2}{\mathfrak{s}} - \sqrt{n}\right| \geq u\right) \leq C \exp(-cu^2)$$

for some constants c and C . Let W_j denote the following variable.

$$W_j = \frac{1}{\|\mathfrak{C}_{k_j} + E_j\|_2} - 1,$$

and let \mathcal{E}_α denote the event

$$\begin{aligned} \mathcal{E}_\alpha &= \cap_{j \in T} \left\{ -\frac{\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)}{1 + \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)} \leq \frac{1}{\|\mathfrak{C}_{k_j} + E_j\|_2} - 1 \right. \\ &\quad \left. \leq \frac{\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)}{1 - \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)} \right\}. \end{aligned}$$

Taking $u = \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)}$, we obtain that

$$\mathbb{P}(\mathcal{E}_\alpha) \geq 1 - \frac{C}{p^\alpha}.$$

On the other hand, we can write $\|A\|_2$ as

$$\|A\|_2 = \left\| \sum_{j \in T} A_j \right\|,$$

where A_j is the matrix

$$A_j = W_j \begin{bmatrix} 0 & \mathfrak{C}_{k_j}^t \beta_j \\ \mathfrak{C}_{k_j} \beta_j & 0 \end{bmatrix}$$

Thus, by the triangular inequality, we have

$$(A.24) \quad \|A\|_2 \leq \left\| \sum_{j \in T} A_j - \mathbb{E}[A_j \mid \mathcal{E}_\alpha] \right\| + \left\| \sum_{j \in T} \mathbb{E}[A_j \mid \mathcal{E}_\alpha] \right\|,$$

and we may apply the Matrix Hoeffding inequality recalled in Appendix B.2. We have that

$$\|A_j\| = |W_j| |\beta_j|$$

which implies that, on \mathcal{E}_α , we have

$$\|A_j - \mathbb{E}[A_j \mid \mathcal{E}_\alpha]\| \leq 2 \frac{\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)}{1 - \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)} |\beta_j|,$$

which, by Assumption 2.7, gives that

$$\|A_j - \mathbb{E}[A_j \mid \mathcal{E}_\alpha]\| \leq 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) |\beta_j|.$$

The Matrix Hoeffding inequality, first applied to the sum and then to its opposite, yields

$$(A.25) \quad \mathbb{P} \left(\left\| \sum_{j \in T} A_j - \mathbb{E}[A_j \mid \mathcal{E}_\alpha] \right\|_2 \geq u \mid \mathcal{E}_\alpha \right) \leq 2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 16 \mathfrak{s}^2 \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)^2 \|\beta_T\|_2^2} \right).$$

On the other hand, we have that

$$\left\| \sum_{j \in T} \mathbb{E}[A_j \mid \mathcal{E}_\alpha] \right\| = \left\| \sum_{j \in T} \mathbb{E}[W_j \mid \mathcal{E}_\alpha] \begin{bmatrix} 0 & \mathfrak{C}_{k_j}^t \beta_j \\ \mathfrak{C}_{k_j} \beta_j & 0 \end{bmatrix} \right\|.$$

Notice that, since $\|\mathfrak{C}_{k_j}\|_2 = 1$, then, $\|\mathfrak{C}_{k_j} + E_j\|_2^2 / \mathfrak{s}^2$ has a noncentral- χ^2 distribution with non-centrality parameter equal to $1/\mathfrak{s}^2$, for all $j \in T$. Thus, we deduce that all the variables $\|\mathfrak{C}_{k_j} + E_j\|_2^2$, $j \in T$, have the same distribution and in particular, the same conditional expectation given \mathcal{E}_α . Therefore,

$$\begin{aligned} \left\| \sum_{j \in T} \mathbb{E}[A_j \mid \mathcal{E}_\alpha] \right\| &= |\mathbb{E}[W_1 \mid \mathcal{E}_\alpha]| \left\| \sum_{j \in T} \begin{bmatrix} 0 & \mathfrak{C}_{k_j}^t \beta_j \\ \mathfrak{C}_{k_j} \beta_j & 0 \end{bmatrix} \right\| \\ &= |\mathbb{E}[W_1 \mid \mathcal{E}_\alpha]| \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2. \end{aligned}$$

But since

$$|\mathbb{E}[W_1 \mid \mathcal{E}_\alpha]| \leq 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right),$$

we obtain

$$\left\| \sum_{j \in T} \mathbb{E}[A_j \mid \mathcal{E}_\alpha] \right\| = 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2.$$

Combining this latter inequality with (A.25), (A.24) becomes

$$\begin{aligned} \mathbb{P} \left(\|A\|_2 \geq 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 + u \mid \mathcal{E}_\alpha \right) \\ \leq 2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 16 \mathfrak{s}^2 \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)^2 \|\beta_T\|_2^2} \right). \end{aligned}$$

Since, for any event \mathcal{A} ,

$$\mathbb{P}(\mathcal{A}) \leq \mathbb{P}(\mathcal{A} \mid \mathcal{E}_\alpha) + \mathbb{P}(\mathcal{E}_\alpha^c),$$

we obtain that

$$\begin{aligned} \mathbb{P} \left(\|A\|_2 \geq 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 + u \right) \\ \leq 2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 16 \mathfrak{s}^2 \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)^2 \|\beta_T\|_2^2} \right) + \frac{C}{p^\alpha}. \end{aligned}$$

Let us now choose u such that

$$2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 16 \mathfrak{s}^2 \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right)^2 \|\beta_T\|_2^2} \right) = \frac{1}{p^\alpha},$$

i.e.

$$u = 8\sqrt{2} \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \|\beta_T\|_2 \sqrt{\alpha \log(p) + \log(2n+2)}.$$

Therefore, we obtain that

$$\begin{aligned} \mathbb{P} \left(\|A\|_2 \geq 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \left(\|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \right. \right. \\ \left. \left. + 8\sqrt{2} \sqrt{\alpha \log(p) + \log(2n+2)} \|\beta_T\|_2 \right) \right) \\ \leq \frac{C+1}{p^\alpha}. \end{aligned} \quad (\text{A.26})$$

Recall that we assumed the β_j associated to the same cluster to have the same sign. Thus, we obtain that

$$\|\beta_T\|_1 = \|\beta_{T^*}^*\|_1 \leq \sqrt{s^*} \|\beta_{T^*}^*\|_2,$$

and using the version of the Invertibility Condition for \mathfrak{C} (3.14), we get

$$\|\beta_T\|_1 = \sqrt{s^* \rho_{\mathfrak{C}}} \|\mathfrak{C}_{\mathcal{K}_T}^* \beta_{T^*}^*\|_2,$$

and thus,

$$\|\beta_T\|_2 = \sqrt{s^* \rho_{\mathfrak{C}}} \|\mathfrak{C}_{\mathcal{K}_T}^* \beta_{T^*}^*\|_2$$

and, using the definition of β_{T^*} ,

$$\|\beta_T\|_2 = \sqrt{s^* \rho_{\mathfrak{C}}} \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2. \quad (\text{A.27})$$

Thus, (A.26) gives

$$\begin{aligned} \mathbb{P} \left(\|A\|_2 \geq 4\mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \left(1 + 8\sqrt{2} \sqrt{\alpha \log(p) + \log(2n+2)} \sqrt{s^* \rho_{\mathfrak{C}}} \right) \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \right) \\ \leq \frac{C+1}{p^\alpha}. \end{aligned} \quad (\text{A.28})$$

A.2. Proof of Lemma 3.4. Recall that

$$\|B\|_2 = \left\| \sum_{j \in T} \frac{\beta_j}{\|\mathfrak{C}_{k_j} + E_j\|_2} E_j \right\|_2.$$

We will use Talagrand's concentration inequality and Dudley's entropy integral bound to study $\|B\|_2$. We start with some preliminary results.

A.2.1. Preliminaries. Let us define the following event:

$$\mathcal{F}_\alpha = \mathcal{E}_\alpha \cap \left\{ \|E_T^t\| \leq \mathfrak{s} \left(\sqrt{n} + \sqrt{s} + \sqrt{2\alpha \log(p)} \right) \right\}.$$

Since E_T^t is i.i.d. with Gaussian entrees $\mathcal{N}(0, \mathfrak{s})$, Theorem B.5 in Appendix B.4 gives

$$\mathbb{P} \left(\|E_T^t\| \geq \mathfrak{s} \left(\sqrt{n} + \sqrt{s} + \sqrt{2\alpha \log(p)} \right) \right) \leq \frac{2}{p^\alpha}.$$

Thus, the union bound gives that $\mathbb{P}(\mathcal{F}_\alpha) \geq 3/p^\alpha$. Let us now turn to the task of bounding $\|B\|_2$.

A.2.2. *Concentration of $\|B\|_2$ using Talagrand's inequality.* Notice that on \mathcal{F}_α , we have

$$\left\| \sum_{j \in T} \frac{\beta_j}{\|\mathfrak{C}_{k_j} + E_j\|_2} E_j \right\|_2 \leq \max_b \left\| \sum_{j \in T} \frac{\beta_j}{b} E_j \right\|_2,$$

where the maximum is over all

$$b \in \left[1 - \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right), 1 + \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p) + \frac{1}{c} \log(s)} \right) \right].$$

Thus, on \mathcal{F}_α ,

$$\|B\|_2 \leq \max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_{i,j} \rangle,$$

the main advantage of this former inequality being that of involving the supremum of a simple Gaussian process. Now, we have

$$\begin{aligned} \mathbb{P} \left(\|B\|_2 - \mathbb{E} \left[\max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle \mid \mathcal{F}_\alpha \right] \geq u \mid \mathcal{F}_\alpha \right) \\ \leq \mathbb{P} \left(\max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle - \mathbb{E} \left[\max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle \mid \mathcal{F}_\alpha \right] \geq u \mid \mathcal{F}_\alpha \right). \end{aligned}$$

Let

$$M_{b,w} = \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle.$$

In order to apply Talagrand's concentration inequality, we have to bound the $M_{b,w}$ on \mathcal{E}_α , and its conditional variance given \mathcal{F}_α . First, by the Cauchy-Schwartz inequality, we have

$$M_{b,w} \leq \frac{1}{b} \|\beta_T\|_2 \sqrt{\sum_{j \in T} (w^t E_j)^2},$$

and thus,

$$\begin{aligned} M_{b,w} &\leq \frac{1}{b} \|\beta_T\|_2 \|E_T^t w\|_2 \\ &\leq \frac{1}{b} \|\beta_T\|_2 \|E_T^t\| \|w\|_2. \end{aligned}$$

Thus, on \mathcal{F}_α , using the fact that $\|w\|_2 = 1$, we have

$$M_{b,w} \leq \mu_{\max} \|\beta_T\|_2,$$

where μ_{\max} is given by (2.6). Let us now turn to the conditional variance of $M_{b,w}$ given \mathcal{F}_α . We have

$$\text{Var}(M_{b,w} \mid \mathcal{F}_\alpha) = \sum_{j \in T} \frac{\beta_j}{b} \text{Var}(E_j^t w \mid \mathcal{F}_\alpha),$$

and, using the Cauchy-Schwartz inequality again, we obtain

$$\text{Var}(M_{b,w} \mid \mathcal{F}_\alpha) = \frac{\|\beta_T\|_2}{b} \sqrt{\sum_{j \in T} \text{Var}^2(E_j^t w \mid \mathcal{F}_\alpha)}.$$

On the other hand, notice that, conditionally on \mathcal{F}_α , $E_j^t w$ is centered. This can easily be seen from the invariance of both the Gaussian law and the event \mathcal{F}_α under the action of orthogonal transformations. Therefore, we have

$$\text{Var}(E_j^t w) \geq \text{Var}(E_j^t w \mid \mathcal{F}_\alpha) \left(1 - \frac{3}{p^\alpha} \right).$$

Moreover, using the fact that $\|w\|_2 = 1$,

$$\text{Var}(E_j^t w) = \mathfrak{s}^2.$$

Therefore,

$$\text{Var}(M_{b,w} \mid \mathcal{F}_\alpha) = \frac{\sqrt{\mathfrak{s}} \mathfrak{s}^2}{b} \|\beta_T\|_2.$$

Using the lower bound on b , we finally obtain

$$\text{Var}(M_{b,w} \mid \mathcal{F}_\alpha) \leq \sigma_{\max}^2 \|\beta_T\|_2,$$

where σ_{\max}^2 is defined by (2.7). With the bound on $M_{b,w}$ and its conditional variance in hand, we are ready to use Talagrand's concentration inequality recalled in Appendix B.5. Thus, Theorem B.6 gives

$$\mathbb{P}\left(\max_{b, \|w\|_2=1} \frac{M_{b,w}}{\mu_{\max} \|\beta_T\|_2} \geq \mathbb{E}\left[\max_{b, \|w\|_2=1} \frac{M_{b,w}}{\mu_{\max} \|\beta_T\|_2} \mid \mathcal{F}_\alpha\right] + \sqrt{2u\gamma} + \frac{u}{3} \mid \mathcal{F}_\alpha\right) \leq \exp(-u),$$

(A.29)

(A.30)

with

$$\gamma = n \frac{\sigma_{\max}^2}{\mu_{\max}^2 \|\beta_T\|_2^2} + \mathbb{E}\left[\max_{b, \|w\|_2=1} \frac{M_{b,w}}{\mu_{\max} \|\beta_T\|_2} \mid \mathcal{F}_\alpha\right].$$

A.2.3. *Control of the conditional expectation of $\max_{b, \|w\|_2=1} \frac{M_{b,w}}{\mu_{\max}}$.* Notice that

$$\mathbb{E}\left[\max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle\right] \geq \mathbb{E}\left[\max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle \mid \mathcal{F}_\alpha\right] \left(1 - \frac{1}{p^\alpha}\right).$$

Therefore, our task boils down to controlling the supremum of a centered gaussian process. For this purpose, let $\tilde{w} = w/b$, which implies that

$$\mathbb{E}\left[\max_{b, \|w\|_2=1} \langle w, \sum_{j \in T} \frac{\beta_j}{b} E_j \rangle\right] = \mathbb{E}\left[\max_{\tilde{w} \in \mathcal{T}} \langle \tilde{w}, \sum_{j \in T} \beta_j E_j \rangle\right]$$

where \mathcal{T} denotes the spherical shell between the sphere centered at zero with radius $r_{\max} = 1 + \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha}{c} \log(p)} + \frac{1}{c} \log(s)\right)$ and the sphere centered at zero with radius $r_{\min} = 2 - r_{\max}$. This can of course be performed using Dudley's entropy bound recalled in Section B.6.1. In the terminology of Section B.6.1, the semi-metric d given by

$$d^2(\tilde{w}, \tilde{w}') = \mathbb{E}\left[\left(\langle \tilde{w} - \tilde{w}', \sum_{j \in T} \beta_j E_j \rangle\right)^2\right].$$

The variables $\beta_j (w - w')^t E_j$, $j \in T$, are centered and have variance equal to $\mathfrak{s}^2 \beta_j^2 \|w - w'\|_2^2$. Thus,

$$d(w, w') = \mathfrak{s} \|\beta_T\|_2 \|w - w'\|_2.$$

Let us now consider the entropy. An upper bound on the covering number of \mathcal{T} with respect to the euclidean distance is given by

$$H(\varepsilon, \mathcal{T}) \leq n \log\left(\frac{3 \mathfrak{s} r_{\max} \|\beta_T\|_2}{\varepsilon}\right).$$

Therefore, by Theorem B.7, we obtain that

$$\mathbb{E}\left[\max_{\tilde{w} \in \mathcal{T}} \langle \tilde{w}, \sum_{j \in T} \beta_j E_j \rangle\right] \leq 12\sqrt{n} \int_0^{\sigma_G} \sqrt{\log\left(\frac{3 \mathfrak{s} r_{\max} \|\beta_T\|_2}{\varepsilon}\right)} d\varepsilon,$$

with

$$\sigma_G = \mathfrak{s} r_{\max} \|\beta_T\|_2.$$

Using the change of variable $\varepsilon' = \frac{\varepsilon}{\mathfrak{s} r_{\max} \|\beta_T\|_2}$, we obtain

$$(A.31) \quad \mathbb{E} \left[\max_{\tilde{w} \in \mathcal{T}} \left\langle \tilde{w}, \sum_{j \in T} \beta_j E_j \right\rangle \right] \leq 12 C_f \mathfrak{s} r_{\max} \|\beta_T\|_2 \sqrt{n},$$

where we recall that

$$C_f = \int_0^3 \sqrt{\log \left(\frac{3}{\varepsilon'} \right)} d\varepsilon'.$$

A.2.4. Conclusion of the proof. To sum up, combining (A.30) and (A.31)

$$(A.32) \quad \mathbb{P} \left(\|B\|_2 \geq 12 C_f \mathfrak{s} \sqrt{n} r_{\max} \|\beta_T\|_2 + \mu_{\max} \|\beta_T\|_2 \left(\sqrt{2u\gamma} + \frac{u}{3} \right) \mid \mathcal{F}_\alpha \right) \leq \exp(-u),$$

with

$$\gamma \leq n \frac{\sigma_{\max}^2}{\mu_{\max}^2 \|\beta_T\|_2^2} + 12 \frac{C_f}{\mu_{\max}} \mathfrak{s} \sqrt{n} r_{\max}.$$

Thus,

$$(A.33) \quad \mathbb{P} \left(\|B\|_2 \geq \left(12 C_f \mathfrak{s} \sqrt{n} r_{\max} + \sqrt{2n\sigma_{\max}^2 \frac{u}{\|\beta_T\|_2^2} + 12 C_f \mu_{\max} \mathfrak{s} \sqrt{n} r_{\max} + \mu_{\max} \frac{u}{3}} \right) \|\beta_T\|_2 \mid \mathcal{F}_\alpha \right) \leq \exp(-u).$$

Taking $u = \alpha \log(p)$ and using Assumption 2.8 gives

$$\mathbb{P} \left(\|B\|_2 \geq \left(12 C_f \mathfrak{s} \sqrt{n} r_{\max} + \alpha \log(p) \mu_{\max} \right) \|\beta_T\|_2 \mid \mathcal{F}_\alpha \right) \leq \frac{1}{p^\alpha}.$$

Using the same trick as before, we have

$$\mathbb{P} \left(\|B\|_2 \geq \left(12 C_f \mathfrak{s} \sqrt{n} r_{\max} + \alpha \log(p) \mu_{\max} \right) \|\beta_T\|_2 \right) \leq \frac{2}{p^\alpha}.$$

Finally, using (A.27), we have

$$\mathbb{P} \left(\|B\|_2 \geq \left(12 C_f \mathfrak{s} \sqrt{n} r_{\max} + \alpha \log(p) \mu_{\max} \right) \sqrt{s^* \rho_{\mathfrak{C}}} \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \right) \leq \frac{2}{p^\alpha}.$$

A.3. Proof of Lemma 3.5. We will use the same arguments based on the Matrix Hoeffding inequality as in A.1. For this purpose, define

$$W_{j^*}^* = \frac{1}{\|\mathfrak{C}_{k_{j^*}} + E_{j^*}\|_2} - 1$$

and wrFite

$$\|A^*\|_2 = \left\| \sum_{j^* \in T^*} W_{j^*}^* \beta_{j^*}^* A_{j^*}^* \right\|.$$

We will need the following lemma.

Lemma A.1. *Let*

$$\mathcal{E}_\alpha^* = \cap_{j^* \in T^*} \left\{ \|E_{j^*}\|_2 \leq \mathfrak{s} \sqrt{n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \right\}.$$

Then, $\mathbb{P}(\mathcal{E}_\alpha^) \geq 1 - p^{-\alpha}$.*

Proof. See Section A.3.2. □

A.3.1. *Control of the deviation of $\|A^*\|_2$ by the Matrix Hoeffding Inequality.* We can write $\|A^*\|_2$ as

$$\|A^*\|_2 = \left\| \sum_{j^* \in T^*} A_{j^*}^* \right\|,$$

where $A_{j^*}^*$ is the matrix

$$A_{j^*}^* = W_{j^*} \begin{bmatrix} 0 & \mathfrak{C}_{k_{j^*}}^t \beta_{j^*}^* \\ \mathfrak{C}_{k_{j^*}} \beta_{j^*}^* & 0 \end{bmatrix}$$

Thus, by the triangular inequality, we have

$$(A.34) \quad \|A^*\|_2 \leq \left\| \sum_{j^* \in T^*} A_{j^*}^* - \mathbb{E}[A_{j^*}^* | \mathcal{E}_\alpha^*] \right\| + \left\| \sum_{j^* \in T^*} \mathbb{E}[A_{j^*}^* | \mathcal{E}_\alpha^*] \right\|,$$

and we may apply the Matrix Hoeffding inequality again. We have that

$$\|A_{j^*}^*\| = |W_{j^*}| |\beta_{j^*}^*|$$

and thus, on \mathcal{E}_α^* ,

$$\|A_{j^*}^* - \mathbb{E}[A_{j^*}^* | \mathcal{E}_\alpha^*]\| \leq 2 \frac{\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}{1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}} |\beta_{j^*}^*|.$$

Under Assumption 2.7, we have

$$\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \leq 0.1,$$

this former inequality becomes

$$\|A_{j^*}^* - \mathbb{E}[A_{j^*}^* | \mathcal{E}_\alpha^*]\| \leq 3\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} |\beta_{j^*}^*|.$$

Applying the Matrix Hoeffding inequality, we obtain

$$(A.35) \quad \begin{aligned} & \mathbb{P} \left(\left\| \sum_{j^* \in T^*} A_{j^*}^* - \mathbb{E}[A_{j^*}^* | \mathcal{E}_\alpha^*] \right\|_2 \geq u \mid \mathcal{E}_\alpha^* \right) \\ & \leq 2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 9 \mathfrak{s}^2 \left(n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \right) \|\beta_{T^*}^*\|_2^2} \right). \end{aligned}$$

Let us now turn to the expectation term, i.e. the last term in (A.34). We have

$$\begin{aligned} \left\| \sum_{j^* \in T^*} \mathbb{E}[A_{j^*}^* | \mathcal{E}_\alpha^*] \right\| &= \left\| \sum_{j^* \in T^*} \mathbb{E}[W_{j^*} | \mathcal{E}_\alpha^*] \begin{bmatrix} 0 & \mathfrak{C}_{k_{j^*}}^t \beta_{j^*}^* \\ \mathfrak{C}_{k_{j^*}} \beta_{j^*}^* & 0 \end{bmatrix} \right\| \\ &\leq \left\| \sum_{j^* \in T^*} 3\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \begin{bmatrix} 0 & \mathfrak{C}_{k_{j^*}}^t \beta_{j^*}^* \\ \mathfrak{C}_{k_{j^*}} \beta_{j^*}^* & 0 \end{bmatrix} \right\|. \end{aligned}$$

This last inequality, when combined with (A.35) and (A.34), implies

$$\begin{aligned} & \mathbb{P} \left(\|A^*\|_2 \geq 3\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \|\mathfrak{C}_{\mathcal{K}_{T^*}} \beta_{T^*}\|_2 + u \mid \mathcal{E}_\alpha^* \right) \\ & \leq 2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 9 \mathfrak{s}^2 \left(n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \right) \|\beta_{T^*}^*\|_2^2} \right), \end{aligned}$$

from which we deduce, by the same trick as in Section A.1, that

$$\begin{aligned} \mathbb{P} \left(\|A^*\|_2 \geq 3\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \|\mathfrak{C}_{\mathcal{K}_{T^*}} \beta_{T^*}\|_2 + u} \right) \\ \leq 2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 9 \mathfrak{s}^2 \left(n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \|\beta_{T^*}^*\|_2^2 \right)} \right) + \frac{1}{p^\alpha}. \end{aligned}$$

Let us now choose u such that

$$2(n+1) \cdot \exp \left(- \frac{u^2}{8 \cdot 9 \mathfrak{s}^2 \left(n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \|\beta_{T^*}^*\|_2^2 \right)} \right),$$

i.e.

$$u = 8\sqrt{2} \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \|\beta_{T^*}^*\|_2 \sqrt{\alpha \log(p) + \log(2n+2)}}.$$

Therefore, we obtain that

$$\begin{aligned} \mathbb{P} \left(\|A^*\|_2 \geq 3\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \|\mathfrak{C}_{\mathcal{K}_{T^*}} \beta_{T^*}\|_2} \right. \\ \left. + 8\sqrt{2} \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \|\beta_{T^*}^*\|_2 \sqrt{\alpha \log(p) + \log(2n+2)}} \right) \\ \leq \frac{2}{p^\alpha}. \end{aligned}$$

By the (3.14), and the definition of β^* , we have

$$\begin{aligned} \|\beta_{T^*}^*\|_2 &\leq \sqrt{\rho \mathfrak{e}} \|\mathfrak{C}_{\mathcal{K}_{T^*}} \beta_{T^*}\|_2, \\ &= \sqrt{\rho \mathfrak{e}} \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \end{aligned}$$

and therefore, we obtain

$$\begin{aligned} \mathbb{P} \left(\|A^*\|_2 \geq 3\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \right. \\ \left. \left(1 + 2\sqrt{2} \sqrt{\rho \mathfrak{e}} \sqrt{\alpha \log(p) + \log(2n+2)} \right) \|\mathfrak{C}_{\mathcal{K}_T} \beta_T\|_2 \right) \\ \leq \frac{2}{p^\alpha}. \end{aligned}$$

A.3.2. *Proof of Lemma A.1.* Using the independence of the E_j , $j \in \mathcal{J}_{k_j^*}$, we have

$$\begin{aligned} \mathbb{P}(\|E_{j^*}\|_2 \geq u) &= \mathbb{P} \left(\min_{j \in \mathcal{J}_{k_j^*}} \|E_j\|_2 \geq u \right) \\ &= \prod_{j \in \mathcal{J}_{k_j^*}} \mathbb{P}(\|E_j\|_2^2 \geq u^2), \\ &\leq \mathbb{P}(\|E_j\|_2^2 \geq u^2)^{\min_{j^* \in T^*} |\mathcal{J}_{k_j^*}|}. \end{aligned}$$

We also have

$$\mathbb{P}(\|E_j\|_2^2 \geq u^2) = 1 - \mathbb{P}(\|E_j\|_2^2 \leq u^2).$$

On the other hand, as is well known, we have

$$\mathbb{P}\left(\frac{\|E_j\|_2^2}{\mathfrak{s}^2} \leq u^2\right) \leq C_\chi \left(\frac{u^2}{n}\right)^n$$

for some positive constant C_χ . Thus, the union bound gives

$$\mathbb{P}\left(\max_{j^* \in T^*} \|E_{j^*}\|_2 \geq u\right) \leq s^* \left(1 - C_\chi \left(\frac{u^2}{n \mathfrak{s}^2}\right)^n\right)^{\min_{j^* \in T^*} |\mathcal{J}_{k_{j^*}}|}.$$

Let us tune u so that

$$s^* \left(1 - C_\chi \left(\frac{u^2}{n \mathfrak{s}^2}\right)^n\right) \leq \frac{1}{p^\alpha}$$

i.e.

$$u^2 \geq \frac{n \mathfrak{s}^2}{C_\chi^{\frac{1}{n}}} \left(1 - (s^* p^{-\alpha})^{\frac{1}{\min_{j^* \in T^*} |\mathcal{J}_{k_{j^*}}|}}\right)^{\frac{1}{n}}$$

and since $\min_{j^* \in T^*} |\mathcal{J}_{k_{j^*}}| \geq \vartheta_* \log(p)^\nu$,

$$u^2 \geq \frac{n \mathfrak{s}^2}{C_\chi^{\frac{1}{n}}} \left(1 - \exp\left(-\frac{\alpha}{\vartheta_* \log(p)^{\nu-1}} - \frac{\log(s^*)}{\vartheta_* \log(p)^\nu}\right)\right)^{\frac{1}{n}}.$$

On $(0, 1)$, we have

$$\exp(-z) \leq 1 - (1 - e^{-1})z$$

and thus,

$$u^2 \geq n \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}},$$

from which the desired estimate follows.

A.4. Proof of Lemma 3.6.

A.4.1. *Concentration of $\|B^*\|_2$.* We start with the concentration of

$$\|B^*\|_2 = \left\| \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{\|\mathfrak{C}_{k_{j^*}} + E_{j^*}\|_2} E_{j^*} \right\|_2.$$

Consider the matrix $E_{T^*}^t$, whose columns are independent. We would like to bound its operator norm.

Lemma A.2. *We have*

$$\mathbb{P}(\|E_{T^*}^t\| \geq \mathfrak{s} K_{n,s^*} \mid \mathcal{E}_\alpha^*) \leq \frac{2}{p^\alpha}$$

where we recall that K_{n,s^*} is defined by (2.8) above.

Proof. See Section A.4.4. □

Define

$$\mathcal{F}_\alpha^* = \mathcal{E}_\alpha^* \cap \{\|E_{T^*}^t\| \leq \mathfrak{s} K_{n,s^*}\}.$$

Thus, the union bound gives that $\mathbb{P}(\mathcal{F}_\alpha^*) \geq 3/p^\alpha$. Let us now turn to the task of bounding $\|B^*\|_2$. Notice that on \mathcal{F}_α^* , we have

$$\left\| \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{\|\mathfrak{C}_{k_{j^*}} + E_{j^*}\|_2} E_{j^*} \right\|_2 \leq \max_b \left\| \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} E_{j^*} \right\|_2,$$

where the maximum is over all

$$b \in \left[1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}}, 1 + \mathfrak{s} \sqrt{n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} \right].$$

Thus, on \mathcal{F}_α^* ,

$$\left\| \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{\|\mathfrak{C}_{k_{j^*}} + E_{j^*}\|_2} E_{j^*} \right\|_2 \leq \max_{b, \|w\|_2=1} \langle w, \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} E_{j^*} \rangle.$$

Now, we have

$$\begin{aligned} & \mathbb{P} \left(\|B^*\|_2 - \mathbb{E} \left[\max_{b, \|w\|_2=1} \langle w, \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} E_{j^*} \rangle \mid \mathcal{F}_\alpha^* \right] \geq u \mid \mathcal{F}_\alpha^* \right) \\ & \leq \mathbb{P} \left(\max_{b, \|w\|_2=1} \langle w, \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} E_{j^*} \rangle - \mathbb{E} \left[\max_{b, \|w\|_2=1} \langle w, \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} E_{j^*} \rangle \mid \mathcal{F}_\alpha^* \right] \geq u \mid \mathcal{F}_\alpha^* \right). \end{aligned}$$

Let

$$M_{b,w}^* = \langle w, \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} E_{j^*} \rangle.$$

We again have to bound $M_{b,w}^*$ on \mathcal{F}_α^* , and its conditional variance given \mathcal{F}_α^* . The Cauchy-Schwartz inequality gives

$$M_{b,w}^* \leq \frac{1}{b} \|\beta_{T^*}^*\|_2 \sqrt{\sum_{j^* \in T^*} (w^t E_{j^*})^2},$$

and thus,

$$\begin{aligned} M_{b,w}^* & \leq \frac{1}{b} \|\beta_{T^*}^*\|_2 \|E_{T^*}^t w\|_2 \\ & \leq \frac{1}{b} \|\beta_{T^*}^*\|_2 \|E_{T^*}^t\| \|w\|_2. \end{aligned}$$

Thus, on \mathcal{F}_α^* , using the fact that $\|w\|_2 = 1$, we have

$$M_{b,w}^* \leq \mu_{\max}^* \|\beta_{T^*}^*\|_2,$$

where

$$\mu_{\max}^* = \mathfrak{s} K_{n,s^*},$$

and K_{n,s^*} is defined by (2.8). Let us now turn to the conditional variance of $M_{b,w}^*$ given \mathcal{F}_α^* .

Lemma A.3. *We have*

$$\text{Var} (M_{b,w}^* \mid \mathcal{F}_\alpha^*) \leq \sigma_{\max}^*{}^2$$

where $\sigma_{\max}^*{}^2$ is defined by (2.9).

Proof. See Section A.4.5. □

Using Talagrand's inequality (Theorem B.6) again, we obtain that

$$\begin{aligned} \text{(A.36)} \quad & \mathbb{P} \left(\max_{b, \|w\|_2=1} \frac{M_{b,w}^*}{\mu_{\max}^* \|\beta_{T^*}^*\|_2} \geq \mathbb{E} \left[\max_{b, \|w\|_2=1} \frac{M_{b,w}^*}{\mu_{\max}^* \|\beta_{T^*}^*\|_2} \mid \mathcal{F}_\alpha^* \right] + \sqrt{2u\gamma^*} + \frac{u}{3} \mid \mathcal{F}_\alpha^* \right) \\ & \leq \exp(-u), \end{aligned}$$

with

$$\gamma^* = n \frac{\sigma_{\max}^*{}^2}{\mu_{\max}^*{}^2 \|\beta_{T^*}^*\|_2^2} + \mathbb{E} \left[\max_{b, \|w\|_2=1} \frac{M_{b,w}^*}{\mu_{\max}^* \|\beta_{T^*}^*\|_2} \mid \mathcal{F}_\alpha^* \right].$$

A.4.2. *Control of the conditional expectation of $\max_{b, \|w\|_2=1} \frac{M_{b,w}^*}{\mu_{\max}^* \|\beta_{T^*}^*\|_2}$.* As in Section A.2, we will use Dudley's entropy integral bound to control the expectation, but this time, the sub-Gaussian version of Section B.9. Let us rewrite

$$\mathbb{E} \left[\max_{b, \|w\|_2=1} M_{b,w}^* \mid \mathcal{F}_\alpha^* \right] = \mathbb{E} \left[\max_{\tilde{w} \in \mathcal{T}^*} \langle \tilde{w}, \sum_{j^* \in T^*} \beta_{j^*}^* E_{j^*} \rangle \mid \mathcal{F}_\alpha^* \right].$$

First, we have to prove the sub-Gaussianity of $M_{b,w}^*$. Notice that, due to rotational invariance of the Gaussian measure, conditionally on \mathcal{F}_α^* , $E_{j^*}^t w$ is centered and

$$\begin{aligned} \mathbb{P} \left(\langle \tilde{w} - \tilde{w}', \sum_{j^* \in T^*} \beta_{j^*}^* E_{j^*} \rangle \geq u \mid \mathcal{F}_\alpha^* \right) &\leq \mathbb{P} \left(\sum_{j^* \in T^*} \beta_{j^*}^* (\tilde{w} - \tilde{w}')^t E_{j^*} \geq u \mid \mathcal{F}_\alpha^* \right) \\ &= \mathbb{P} \left(\sum_{j^* \in T^*} \beta_{j^*}^* (O_{\tilde{w}-\tilde{w}'} D(\zeta) E_{j^*})^t (\tilde{w} - \tilde{w}') \geq u \mid \mathcal{F}_\alpha^* \right). \end{aligned}$$

where ζ is a rademacher ± 1 random vector, $O_{\tilde{w}-\tilde{w}'}$ is the orthogonal transform which sends $\tilde{w} - \tilde{w}'$ to the vector $\|\tilde{w} - \tilde{w}'\|_2 / \sqrt{n} e$, where e is the vector of all ones. Thus,

$$\mathbb{P} \left(\langle \tilde{w} - \tilde{w}', \sum_{j^* \in T^*} \beta_{j^*}^* E_{j^*} \rangle \geq u \mid \mathcal{F}_\alpha^* \right) = \mathbb{P} \left(\frac{\|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{j^* \in T^*} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \geq u \mid \mathcal{F}_\alpha^* \right).$$

We now study the sub-Gaussianity of $\sum_{i=1}^n \zeta_i E_{i,j^*}$. Using the Laplace transform version of Hoeffding's inequality, we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(\eta \frac{\|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{i=1}^n \zeta_i \beta_{j^*}^* E_{i,j^*} \right) \mid E_{j^*}, \mathcal{F}_\alpha^* \right] \\ \leq \exp \left(\eta^2 \frac{\|\tilde{w} - \tilde{w}'\|_2^2}{n} \beta_{j^*}^{*2} \mathfrak{s}^2 \left(n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \right) \right). \end{aligned}$$

Therefore, using independence of the E_{j^*} 's, we have that

$$\begin{aligned} \mathbb{E} \left[\exp \left(\frac{\eta \|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{j^* \in T^*} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \right) \mid \mathcal{F}_\alpha^* \right] \\ = \mathbb{E} \left[\mathbb{E} \left[\exp \left(\frac{\eta \|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{j^* \in T^*} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \right) \mid E_{j^*}, \mathcal{F}_\alpha^* \right] \mid \mathcal{F}_\alpha^* \right] \\ = \mathbb{E} \left[\prod_{j^* \in T^*} \mathbb{E} \left[\exp \left(\frac{\eta \|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \right) \mid E_{j^*}, \mathcal{F}_\alpha^* \right] \mid \mathcal{F}_\alpha^* \right] \\ \leq \exp \left(\eta^2 \|\tilde{w} - \tilde{w}'\|_2^2 \|\beta_{T^*}^*\|_2^2 \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \right). \end{aligned}$$

Now Chernov's bound gives

$$\begin{aligned}
& \mathbb{P} \left(\frac{\|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{j^* \in \mathcal{T}^*} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \geq u \mid \mathcal{F}_\alpha^* \right) \\
& \leq e^{-\eta u} \mathbb{E} \left[\exp \left(\frac{\eta \|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{j^* \in \mathcal{T}^*} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \right) \mid \mathcal{F}_\alpha^* \right] \\
& \leq \exp \left(\eta^2 \|\tilde{w} - \tilde{w}'\|_2^2 \|\beta_{T^*}^*\|_2^2 \mathfrak{s}^2 \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} - \eta u \right).
\end{aligned}$$

Optimizing in η gives

$$\begin{aligned}
& \mathbb{P} \left(\frac{\|\tilde{w} - \tilde{w}'\|_2}{\sqrt{n}} \sum_{j^* \in \mathcal{T}^*} \beta_{j^*}^* \sum_{i=1}^n \zeta_i E_{i,j^*} \geq u \mid \mathcal{F}_\alpha^* \right) \\
& \leq \exp \left(-\frac{1}{4} \frac{u^2}{\|\tilde{w} - \tilde{w}'\|_2^2 \|\beta_{T^*}^*\|_2^2 \mathfrak{s}^2 \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \right).
\end{aligned}$$

Using the union bound and invariance of the bound with respect to sign change, we thus obtain

$$\begin{aligned}
& \mathbb{P} \left(\left| \langle \tilde{w} - \tilde{w}', \sum_{j^* \in \mathcal{T}^*} \beta_{j^*}^* E_{j^*} \rangle \right| \geq u \mid \mathcal{F}_\alpha^* \right) \\
& \leq 2 \exp \left(-\frac{1}{4} \frac{u^2}{\|\tilde{w} - \tilde{w}'\|_2^2 \|\beta_{T^*}^*\|_2^2 \mathfrak{s}^2 \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \right).
\end{aligned}$$

Thus, the process is sub-Gaussian with the semi-metric d , given by

$$d^2(\tilde{w}, \tilde{w}') = 4 \|\tilde{w} - \tilde{w}'\|_2^2 \|\beta_{T^*}^*\|_2^2 \mathfrak{s}^2 \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}.$$

Let us now apply Theorem B.9. The diameter of \mathcal{T}^* is bounded from above by

$$r_{\max}^* = \frac{1}{1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}.$$

An upper bound on the covering number of \mathcal{T}^* with respect to the semi-metric d is given by

$$H(\varepsilon, \mathcal{T}^*) \leq n \log \left(\frac{3 \cdot 2 r_{\max}^* \|\beta_{T^*}^*\|_2 \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}{\varepsilon} \right).$$

Therefore, by Theorem B.7, we obtain that

$$\mathbb{E} \left[\max_{\tilde{w} \in \mathcal{T}^*} \left\langle \tilde{w}, \sum_{j \in T} \beta_j E_j \right\rangle \mid \mathcal{F}_\alpha^* \right] \leq 12\sqrt{n} \int_0^{\sigma_G} \sqrt{\log \left(\frac{6 r_{\max}^* \|\beta_{T^*}^*\|_2 \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}{\varepsilon} \right)} d\varepsilon,$$

with

$$\sigma_G = r_{\max}^*.$$

Using the change of variable

$$\varepsilon' = \frac{\varepsilon}{2 r_{\max}^* \|\beta_{T^*}^*\|_2 \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}}},$$

we obtain

$$(A.37) \quad \mathbb{E} \left[\max_{\tilde{w} \in \mathcal{T}} \langle \tilde{w}, \sum_{j \in T} \beta_j E_j \rangle \mid \mathcal{F}_\alpha^* \right] \leq 24 r_{\max}^* \|\beta_{T^*}^*\|_2 \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^*,$$

where

$$C_f^* = \int_0^3 \sqrt{\log\left(\frac{3}{\varepsilon'}\right)} d\varepsilon'.$$

A.4.3. *Last step of the proof.* Combining (A.36) and (A.37), we obtain

$$(A.38) \quad \mathbb{P} \left(\|B^*\| \geq 24 r_{\max}^* \|\beta_{T^*}^*\|_2 \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^* + \mu_{\max}^* \|\beta_{T^*}^*\|_2 \left(\sqrt{2u\gamma^*} + \frac{u}{3} \right) \mid \mathcal{F}_\alpha^* \right) \leq \exp(-u),$$

with

$$\gamma^* = n \frac{\sigma_{\max}^{*2}}{\mu_{\max}^*{}^2 \|\beta_{T^*}^*\|_2^2} + \mathbb{E} \left[\max_{b, \|w\|_2=1} \frac{M_{b,w}^*}{\mu_{\max}^* \|\beta_{T^*}^*\|_2} \mid \mathcal{F}_\alpha^* \right].$$

Therefore, taking $u = \alpha \log(p)$ we have

$$\mathbb{P} \left(\|B^*\| \geq 24 r_{\max}^* \|\beta_{T^*}^*\|_2 \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^* + \left(\sqrt{2\alpha \log(p)} (L^*) + \frac{\alpha \log(p)}{3} \mu_{\max}^* \right) \|\beta_{T^*}^*\|_2 \mid \mathcal{F}_\alpha^* \right) \leq p^{-\alpha},$$

with

$$L^* = n \frac{\sigma_{\max}^{*2}}{\|\beta_{T^*}^*\|_2^2} + 24 \mu_{\max}^* r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^*.$$

Using Assumption 2.9, we obtain

$$\mathbb{P} \left(\|B^*\| \geq \left(24 r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^* + \mu_{\max}^* \alpha \log(p) \right) \|\beta_{T^*}^*\|_2 \mid \mathcal{F}_\alpha^* \right) \leq p^{-\alpha}.$$

Using the same trick as before, we obtain

$$\mathbb{P} \left(\|B^*\| \geq \left(24 r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha (1-e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^* + \mu_{\max}^* \alpha \log(p) \right) \|\beta_{T^*}^*\|_2 \right) \leq \frac{2}{p^\alpha}.$$

Notice further that

$$\begin{aligned}\|\beta_{T^*}^*\|_2 &\leq (1 + \rho_{\mathfrak{C}}) \|\mathfrak{C}_{\mathcal{K}_{T^*}} \beta_{T^*}^*\|_2 \\ &= (1 + \rho_{\mathfrak{C}}) \|\mathfrak{C}_T \beta_T\|_2,\end{aligned}$$

by definition of β^* . Thus, we obtain that

$$\begin{aligned}\mathbb{P}\left(\|B^*\| \geq \left(24 r_{\max}^* \mathfrak{s} \sqrt{\left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}} C_f^* \right. \right. \\ \left. \left. + \mu_{\max}^* \alpha \log(p)\right) \sqrt{\rho_{\mathfrak{C}}} \|\mathfrak{C}_T \beta_T\|_2\right) \leq \frac{2}{p^\alpha}.\end{aligned}$$

as desired.

A.4.4. *Proof of Lemma A.2.* Let us first notice that since $\|E_{T^*}\| = \|E_{T^*}^t\|$, we can write

$$\begin{aligned}\|E_{T^*}^t\| &= \sqrt{\|E_{T^*} E_{T^*}^t\|} \\ &= \sqrt{\left\| \sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \right\|}\end{aligned}$$

This latter expression is well suited for our problem, since it is the norm of the sum of independent positive semi-definite random matrices, for which the Matrix Chernov inequality of Section B.3 applies. In order to apply this inequality, we need a bound on the norm of each summand. By Lemma A.1, on \mathcal{E}^* , we have

$$\begin{aligned}\|E_{j^*} E_{j^*}^t\| &= \|E_{j^*}\|_2^2 \\ &\leq \mathfrak{s}^2 n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}.\end{aligned}$$

We also need a bound on the norm of the expectation. We have

$$\left\| \mathbb{E} \left[\sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \mid \mathcal{F}_\alpha^* \right] \right\| = \left\| \sum_{j^* \in T^*} \mathbb{E} [E_{j^*} E_{j^*}^t \mid \mathcal{F}_\alpha^*] \right\|.$$

Due to rotational invariance, we have that the law of E_{j^*} is the same as the law of $D(\zeta)E_{j^*}$, where ζ_1, \dots, ζ_n are i.i.d. Rademacher ± 1 random variables independent from E_{j^*} . Thus,

$$\mathbb{E} [\zeta_i E_{i,j^*} \zeta_{i'} E_{i',j^*} \mid \mathcal{E}_\alpha^*] = \mathbb{E} [\mathbb{E} [\zeta_i E_{i,j^*} \zeta_{i'} E_{i',j^*} \mid E_{i,j^*}, E_{i',j^*} \mid \mathcal{E}_\alpha^*]]$$

$$(A.39) \quad = 0.$$

On the other hand, we have the following result.

Lemma A.4. *We have*

$$\mathbb{E} [E_{i,j^*}^2 \mid \mathcal{E}_\alpha^*] \leq \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi}\right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}}\right)^{\frac{1}{n}}.$$

Proof. Due to rotational invariance of the law of E_{j^*} and the event \mathcal{E}_α^* , we have

$$\mathbb{E} [E_{1,j^*}^2 \mid \mathcal{E}_\alpha^*] = \dots = \mathbb{E} [E_{n,j^*}^2 \mid \mathcal{E}_\alpha^*].$$

Therefore,

$$\mathbb{E} [E_{i,j^*}^2 \mid \mathcal{E}_\alpha^*] \leq \frac{1}{n} \mathbb{E} \left[\sum_{i'=1}^n E_{i',j^*}^2 \mid \mathcal{E}_\alpha^* \right]$$

and by the definition of \mathcal{E}_α^* ,

$$\mathbb{E}[E_{i,j^*}^2 | \mathcal{E}_\alpha^*] = \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}.$$

□

Based on this lemma, and the fact that the matrix

$$\mathbb{E} \left[\sum_{j^* \in T^*} E_{j^*} E_{j^*}^t | \mathcal{F}_\alpha^* \right],$$

is diagonal by (A.39), we obviously obtain that

$$\left\| \mathbb{E} \left[\sum_{j^* \in T^*} E_{j^*} E_{j^*}^t | \mathcal{F}_\alpha^* \right] \right\| = \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}.$$

With the bound on the norm of the expectation and on the variance in hand, we are now ready to apply the Matrix Chernov inequality and obtain

$$\begin{aligned} \mathbb{P} \left(\left\| \sum_{j^* \in T^*} E_{j^*} E_{j^*}^t \right\| \geq u | \mathcal{F}_\alpha^* \right) \\ \leq n \left(\frac{e \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}{u} \right)^{\frac{u}{\mathfrak{s}^2 n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}. \end{aligned}$$

Let us finally tune u so that the right hand side term is less than $p^{-\alpha}$, i.e.

$$\begin{aligned} \log(n) + \log \left(\frac{e \mathfrak{s}^2 \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}{u} \right) \\ \leq -\alpha \frac{\mathfrak{s}^2 n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}{u} \log(p). \end{aligned}$$

Take

$$(A.40) \quad u = \alpha \mathfrak{s}^2 n \left(\frac{\alpha (1 - e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}} \log(p).$$

Since, by assumption, $p \geq e^{2 - \log(\alpha)}$, we have $-\log(\log(p)) + \log(e/\alpha) \leq -1$. Moreover, the value of u given by (A.40) is less than or equal to $\mathfrak{s}^2 K_{n,s^*}^2$ with K_{n,s^*} given by (2.8). This completes the proof.

A.4.5. *Proof of Lemma A.3.* Independence of the E_{j^*} , $j^* \in T^*$ allows to write

$$\text{Var}(M_{b,w}^* | \mathcal{F}_\alpha^*) = \sum_{j^* \in T^*} \frac{\beta_{j^*}^*}{b} \text{Var}(E_{j^*}^t w | \mathcal{F}_\alpha^*),$$

and, using the Cauchy-Schwartz inequality again, we obtain

$$\text{Var}(M_{b,w}^* | \mathcal{F}_\alpha^*) = \frac{\|\beta_{T^*}^*\|_2}{b} \sqrt{\sum_{j^* \in T^*} \text{Var}^2(E_{j^*}^t w | \mathcal{F}_\alpha^*)}.$$

On the other hand, notice that, due to rotational invariance of the Gaussian measure, conditionally on \mathcal{F}_α^* , $E_{j^*}^t w$ is centered and

$$\begin{aligned} \text{Var}(E_{j^*}^t w | \mathcal{F}_\alpha^*) &= \mathbb{E} \left[((O_w D(\zeta) E_{j^*})^t w)^2 | \mathcal{F}_\alpha^* \right], \\ &= \mathbb{E} \left[(E_{j^*}^t D(\zeta) O_w w)^2 | \mathcal{F}_\alpha^* \right], \end{aligned}$$

where ζ is a rademacher ± 1 random vector, O_w is the orthogonal transform which sends w to the vector $1/\sqrt{n}e$, where e is the vector of all ones. Thus,

$$\text{Var} (E_{j*}^t w \mid \mathcal{F}_\alpha^*) = \frac{1}{n} \mathbb{E} \left[\mathbb{E} \left[(E_{j*}^t D(\zeta) e)^2 \mid E, \mathcal{F}_\alpha^* \right] \mid \mathcal{F}_\alpha^* \right],$$

Moreover,

$$\mathbb{E} \left[(E_{j*}^t D(\zeta) e)^2 \mid E, \mathcal{F}_\alpha^* \right] = \mathbb{E} \left[\left(\sum_{i=1}^n E_{i,j*} \zeta_i \right)^2 \mid E, \mathcal{F}_\alpha^* \right]$$

and expanding the square of the sum gives

$$\mathbb{E} \left[(E_{j*}^t D(\zeta) e)^2 \mid E, \mathcal{F}_\alpha^* \right] = \|E_{j*}\|_2^2.$$

Using the bound on b , we finally obtain

$$\text{Var} (M_{b,w}^* \mid \mathcal{F}_\alpha^*) \leq \sigma_{\max}^*{}^2,$$

where $\sigma_{\max}^*{}^2$ is given by (2.9).

APPENDIX B. NORMS OF RANDOM MATRICES, ε -NETS AND CONCENTRATION INEQUALITIES

B.1. Norms and coverings.

Proposition B.1. ([22, Proposition 2.1]). *For any positive integer d , there exists an ε -net of the unit sphere of \mathbb{R}^d of cardinality*

$$2d \left(1 + \frac{2}{\varepsilon} \right)^{d-1} \leq \left(\frac{3}{\varepsilon} \right)^d.$$

The next proposition controls the approximation of the norm based on an ε -net.

Proposition B.2. ([22, Proposition 2.2]). *Let \mathcal{N} be an ε -net of the unit sphere of \mathbb{R}^d and let \mathcal{N}' be an ε' -net of the unit sphere of $\mathbb{R}^{d'}$. Then for any linear operator $A : \mathbb{R}^d \mapsto \mathbb{R}^{d'}$, we have*

$$\|A\| \leq \frac{1}{(1-\varepsilon)(1-\varepsilon')} \sup_{\substack{v \in \mathcal{N} \\ w \in \mathcal{N}'}} |v^t A w|.$$

B.2. The Matrix Hoeffding Inequality. A Non-commutative version of the famous Hoeffding inequality was proposed in [25]. We recall this result for convenience.

Theorem B.3. *Consider a finite sequence $(U_j)_{j \in T}$ of independent random, self-adjoint matrices with dimension d , and let $(U_j)_{j \in T}$ be a sequence of deterministic self-adjoint matrices. Assume that each random matrix satisfies*

$$\mathbb{E}[U_j] = 0 \quad \text{and} \quad U_j^2 \preceq V_j^2 \quad \text{a.s.}$$

for all $j \in T$. Then, for all $u \geq 0$,

$$\mathbb{P} \left(\lambda_{\max} \left(\sum_{j \in T} U_j \right) \geq t \right) \leq d \cdot \exp \left(- \frac{u^2}{8 \left\| \sum_{j \in T} V_j^2 \right\|} \right).$$

B.3. The Matrix Chernov inequality. The following non-commutative version of Chernoff's inequality was recently established in [25].

Theorem B.4. (Matrix Chernoff Inequality [25]) *Let X_1, \dots, X_p be independent random positive semi-definite matrices taking values in $\mathbb{R}^{d \times d}$. Set $S_p = \sum_{j=1}^p X_j$. Assume that for all $j \in \{1, \dots, p\}$ $\|X_j\| \leq B$ a.s. and*

$$\|\mathbb{E} S_p\| \leq \mu_{\max}.$$

Then, for all $r \geq e \mu_{\max}$,

$$\mathbb{P} (\|S_p\| \geq r) \leq d \left(\frac{e \mu_{\max}}{r} \right)^{r/B}.$$

(Set $r = (1 + \delta) \mu_{\max}$ and use $e^\delta \leq e^{1+\delta}$ in Theorem 1.1 [25].)

B.4. Gaussian i.i.d. matrices. The following result on random matrices with Gaussian i.i.d. entries can be found in [27, Corollary 5.35].

Theorem B.5. *Let G be an $n \times m$ matrix whose entries are independent standard normal random variables. Then for every $u \geq 0$, with probability at least $1 - 2 \exp(-u^2/2)$, one has*

$$\sqrt{n} - \sqrt{m} - u \leq \sigma_{\min}(G) \leq \sigma_{\max}(G) \leq \sqrt{n} + \sqrt{m} + u.$$

B.5. Talagrand's concentration inequality for empirical processes. The following theorem, which is a version of Talagrand's concentration inequality for empirical processes, was proved in [4, Theorem 2.3].

Theorem B.6. *Let X_i be a sequence of i.i.d. variables taking values in a Polish space \mathcal{X} , and let \mathcal{F} be a countable family of functions from \mathcal{X} to \mathbb{R} and assume that all functions f in \mathcal{F} are measurable, square integrable and satisfy $\mathbb{E}[f] = 0$. If $\sup_{f \in \mathcal{F}} \text{ess sup } f \leq 1$, then we denote*

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^n f(X_i).$$

Let σ_{\max} be a positive number such that $\sigma_{\max}^2 \geq \sup_{f \in \mathcal{F}} \text{Var}(f(X_1))$ almost surely, then, for all $u \geq 0$, we have

$$\mathbb{P}\left(Z \geq \mathbb{E}[Z] + \sqrt{2u\gamma} + \frac{u}{3}\right) \leq \exp(-u),$$

with $\gamma = n\sigma_{\max}^2 + \mathbb{E}[Z]$.

B.6. Dudley's entropy integral bound. Let (\mathcal{T}, d) denote a semi-metric space and denote by $H(\delta, \mathcal{T})$ the δ -entropy number of (\mathcal{T}, d) for all positive real number δ .

B.6.1. The Gaussian case. Let $(G_t)_{t \in \mathcal{T}}$ be a centered gaussian process indexed by \mathcal{T} and set d to be the covariance pseudo-metric defined by

$$d(t, t') = \sqrt{\mathbb{E}[(G_t - G_{t'})^2]}.$$

Then, we have the following important theorem of Dudley, which can be found in the present form in [18].

Theorem B.7. *Assume that (\mathcal{T}, d) is totally bounded. If $\sqrt{H(\delta, \mathcal{T})}$ is integrable at zero, then*

$$\mathbb{E}\left[\sup_{t \in \mathcal{T}} G_t\right] \leq 12 \int_0^{\sigma_G} \sqrt{H(\delta, \mathcal{T})} d\delta,$$

where

$$\sigma_G^2 = \sup_{t \in \mathcal{T}} \mathbb{E}[G_t^2].$$

B.6.2. The sub-Gaussian case. We start with the definition of sub-Gaussian processes.

Definition B.8. *A centered process $(S_t)_{t \in \mathcal{T}}$ is said to be sub-Gaussian if for all $(t, t') \in \mathcal{T}^2$, and for all $u > 0$,*

$$\mathbb{P}(|X_t - X_{t'}| \geq u) \leq 2 \exp\left(-\frac{u^2}{d^2(t, t')}\right).$$

One easily checks that a Gaussian process is sub-Gaussian with the covariance semi-metric in the former definition. Let $(S_t)_{t \in \mathcal{T}}$ be a centered sub-Gaussian process. We then have the following standard result.

Theorem B.9. *Assume that (\mathcal{T}, d) is totally bounded. If $\sqrt{H(\delta, \mathcal{T})}$ is integrable at zero, then*

$$\mathbb{E}\left[\sup_{t \in \mathcal{T}} S_t\right] = C_{\text{chain}} \int_0^{\text{diam}(\mathcal{T})} \sqrt{H(\delta, \mathcal{T})} d\delta$$

for some positive constant C_{chain} .

APPENDIX C. VERIFYING THE CANDES-PLAN CONDITIONS

The goal of this section is to Proposition 3.1 which gives a version of Candès and Plan's conditions adapted to our Gaussian mixture model.

C.1. Important properties of \mathfrak{C} . The invertibility condition for (3.14) is a direct consequence of [24]. An alternative approach, based on the Matrix Chernov inequality is proposed in [13], with improved constants. We have in particular

Theorem C.1. [13, Theorem 1] *Let $r \in (0, 1)$, $\alpha \geq 1$. Let Assumptions 2.2 and 2.4 hold with*

$$(C.41) \quad C_{spar} \geq \frac{r^2}{4(1+\alpha)e^2}.$$

With $\mathcal{K} \subset \{1, \dots, K\}$ chosen randomly from the uniform distribution among subsets with cardinality s^ , the following bound holds:*

$$(C.42) \quad \mathbb{P}(\|\mathfrak{C}_{\mathcal{K}}^t \mathfrak{C}_{\mathcal{K}} - \text{Id}_s\| \geq r) \leq \frac{216}{p^\alpha}.$$

Moreover, the following property will also be very useful.

Lemma C.2. (Adapted from [13, Lemma 5.3]) *If $v^2 \geq e s^* \|\mathfrak{C}\|/K_o$, we have*

$$\mathbb{P}\left(\max_{k \in \mathcal{K}^c} \|\mathfrak{C}_{\mathcal{K}}^t \mathfrak{C}_k\| \geq \frac{v}{1-r}\right) \leq K_o \left(e \frac{s^* \|\mathfrak{C}\|^2}{K_o v^2}\right)^{\frac{v^2}{\mu(\mathfrak{C})^2}}.$$

Based on this lemma, we easily get the following bound.

Lemma C.3. *Take $C_{col} \geq e^2(\alpha+1) \max\{\sqrt{C_{spar}}, C_\mu\}/(1-r)$. Then, we have*

$$\mathbb{P}\left(\max_{k \in \mathcal{K}^c} \|\mathfrak{C}_{\mathcal{K}}^t \mathfrak{C}_k\| \geq \frac{C_{col}}{(1-r)\sqrt{\log(p)}}\right) \leq \frac{1}{p^\alpha}.$$

Proof. Taking $v = C_{col}/\sqrt{\log(p)}$, we obtain from Lemma C.2

$$\mathbb{P}\left(\max_{k \in \mathcal{K}^c} \|\mathfrak{C}_{\mathcal{K}}^t \mathfrak{C}_k\| \geq \frac{C_{col}}{(1-r)\sqrt{\log(p)}}\right) \leq K_o \left(e \frac{s^* \|\mathfrak{C}\|^2 \log(p)}{K_o C_{col}^2}\right)^{\frac{C_{col}^2}{C_\mu^2} \log(p)}.$$

Using (2.4), this gives

$$\mathbb{P}\left(\max_{k \in \mathcal{K}^c} \|\mathfrak{C}_{\mathcal{K}}^t \mathfrak{C}_k\| \geq \frac{C_{col}}{(1-r)\sqrt{\log(p)}}\right) \leq K_o \left(e \frac{C_{spar}}{C_{col}^2}\right)^{\frac{C_{col}^2}{C_\mu^2} \log(p)}.$$

Since $C_{col} \geq e^2(\alpha+1) \max\{\sqrt{C_{spar}}, C_\mu\}$, we get

$$K_o \left(e \frac{C_{spar}}{C_{col}^2}\right)^{\frac{C_{col}^2}{C_\mu^2} \log(p)} \leq K_o \left(\frac{e}{\alpha+1}\right)^{(\alpha+1) \log(p)}$$

and since, by Assumption 2.1, $K_o \leq p$, we obtain that

$$\mathbb{P}\left(\max_{k \in \mathcal{K}^c} \|\mathfrak{C}_{\mathcal{K}}^t \mathfrak{C}_k\| \geq \frac{C_{col}}{(1-r)\sqrt{\log(p)}}\right) \leq \frac{1}{p^\alpha}.$$

□

C.2. Similar properties for X_{T^*} .

C.2.1. *Control of $\|X_{T^*}^t X_{T^*} - I\|$.* We have

$$\sigma_{\min}(X_{T^*}^t X_{T^*}) = \sigma_{\min}\left((\mathfrak{C}_{\mathcal{K}_{T^*}} + E_{T^*})^t D_*^2 (\mathfrak{C}_{\mathcal{K}_{T^*}} + E_{T^*})\right)$$

where (see Step 1 in the proof of Proposition 3.2) D_* is a diagonal matrix whose diagonal elements are indexed by T^* and are defined by

$$D_{*,j^*,j^*} = \frac{1}{\|\mathfrak{C}_{k_{j^*}} + E_{j^*}\|_2},$$

for $j^* \in T^*$. By the definition of \mathcal{E}_α^* , we have

$$\sigma_{\min}(D_*) \geq \frac{1}{1 + \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}.$$

and

$$\sigma_{\max}(D_*) \leq \frac{1}{1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}.$$

By the triangular inequality,

$$\begin{aligned} \sigma_{\min}(X_{T^*}^t X_{T^*}) &\geq \sigma_{\min}(\mathfrak{C}_{\mathcal{K}}^t D_*^2 \mathfrak{C}_{\mathcal{K}}) - \|\mathfrak{C}_{\mathcal{K}}^t D_*^2 E_{T^*}\| - \|E_{T^*}^t D_*^2 E_{T^*}\| \\ &\geq \frac{1-r}{\left(1 + \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}\right)^2} \\ &\quad - \frac{(1+r) \|E_{T^*}\| + \|E_{T^*}\|^2}{\left(1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}\right)^2}. \end{aligned}$$

and

$$\begin{aligned} \sigma_{\max}(X_{T^*}^t X_{T^*}) &\leq \|\mathfrak{C}_{\mathcal{K}}^t D_*^2 \mathfrak{C}_{\mathcal{K}}\| + \|\mathfrak{C}_{\mathcal{K}}^t D_*^2 E_{T^*}\| + \|E_{T^*}^t D_*^2 E_{T^*}\| \\ &\leq \frac{(1+r) + (1+r) \|E_{T^*}\| + \|E_{T^*}\|^2}{\left(1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}\right)^2}. \end{aligned}$$

Moreover, using Theorem C.1 and Lemma A.2, we obtain

$$\mathbb{P}(\|X_{T^*}^t X_{T^*} - I\| \geq r^* \mid \mathcal{E}_\alpha^*) \leq \frac{218}{p^\alpha}$$

with r^* given by

$$\begin{aligned} (C.43) \quad r^* &= \max \left\{ \frac{(1+r) + (1+r) \mathfrak{s} K_{n,s^*} + \mathfrak{s}^2 K_{n,s^*}^2}{\left(1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}\right)^2} - 1; \right. \\ &\quad 1 - \left(\frac{1-r}{\left(1 + \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}\right)^2} \right. \\ &\quad \left. \left. - \frac{(1+r) \mathfrak{s} K_{n,s^*} + \mathfrak{s}^2 K_{n,s^*}^2}{\left(1 - \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}\right)^2} \right) \right\}. \end{aligned}$$

Using Assumption (2.7), we have

$$\mathfrak{s} K_{n,s^*} \leq C_{\mathfrak{s},n,p} \frac{\sqrt{\alpha \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}{\left(1 + \sqrt{\frac{\alpha+1}{c} \log(p)} \right)},$$

and thus, by Assumption 2.7,

$$\begin{aligned} \mathfrak{s} K_{n,s^*} &\leq C_{\mathfrak{s},n,p} \sqrt{\alpha \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}, \\ &\leq 0.1 \cdot r. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} &\leq \frac{C_{\mathfrak{s},n,p}}{\sqrt{\log(p)}} \frac{\sqrt{\left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}}}{\left(1 + \sqrt{\frac{\alpha+1}{c} \log(p)} \right)} \\ &\leq \frac{C_{\mathfrak{s},n,p}}{\sqrt{\log(p)}} \sqrt{\left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \end{aligned}$$

which, by Assumption 2.7, gives

$$\mathfrak{s} \sqrt{n \left(\frac{\alpha(1-e^{-1})}{\vartheta_* C_\chi} \right)^{\frac{1}{n}} \left(\frac{1}{\log(p)^{\nu-1}} \right)^{\frac{1}{n}}} \leq \frac{0.1 \cdot r}{\log(p)}.$$

Summing up, we get

$$\begin{aligned} r^* &\leq \frac{(1+r) + (1+r) \frac{0.1 \cdot r + 0.01 \cdot r^2}{\left(1 - \frac{0.1 \cdot r}{\log(p)} \right)^2}}{1} - 1 \\ &\leq (1.1 \cdot r + 0.11 \cdot r^2) \\ &\quad + 2 \left(1 + 1.1 \cdot r + 0.11 \cdot r^2 \right) \frac{0.1 \cdot r}{\log(p)} \end{aligned}$$

and by Assumption 2.10,

$$r^* \leq 1.1 \cdot r (1.1 + 0.11 \cdot r).$$

Thus, using Lemma A.1,

$$\mathbb{P}(\|X_{T^*}^t X_{T^*} - I\| \geq 1.1 \cdot r(1.1 + 0.11 \cdot r)) \leq \frac{218 + 1}{p^\alpha},$$

C.2.2. *Control of $\max_{k \in T^{*c}} \|X_{T^*}^t X_k\|_2$.* By the triangular inequality, we have that

$$\begin{aligned} \max_{k \in T^{*c}} \|X_{T^*}^t X_k\|_2 &= \max_{k \in T^{*c}} \|(\mathfrak{C}_K + E_{T^*})^t D_*^2 (\mathfrak{C}_k + E_k)\|_2 \\ &\leq \left(\max_{k \in T^{*c}} \|\mathfrak{C}_K^t \mathfrak{C}_k\| + \|\mathfrak{C}_K\| \max_{k \in T^{*c}} \|E_k\|_2 \right. \\ &\quad \left. + \|E_{T^*}\| \max_{k \in T^{*c}} \|E_k\|_2 \right) \|D_*\|^2. \end{aligned}$$

A computation analogous to the one for the probability of \mathcal{E}_α gives that

$$\mathbb{P} \left(\max_{k \in \{1, \dots, p\}} \|E_k\|_2 \geq \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha+1}{c} \log(p)} \right) \mid \mathcal{E}_\alpha^* \right) \leq \frac{C}{p^\alpha}.$$

Thus, using Lemma C.3 and Lemma A.2, we obtain

$$\mathbb{P} \left(\max_{k \in T^{*c}} \|X_{T^*}^t X_k\|_2 \geq \frac{C_{col}}{\sqrt{\log(p)}} + (1 + r + \mathfrak{s}K_{n,s^*}) \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha+1}{c} \log(p)} \right) \mid \mathcal{E}_\alpha^* \right) \leq \frac{C+2}{p^\alpha}.$$

Since, by Assumption (2.7),

$$(1 + r + \mathfrak{s}K_{n,s^*}) \mathfrak{s} \left(\sqrt{n} + \sqrt{\frac{\alpha+1}{c} \log(p)} \right) \leq (1 + 1.1 \cdot r) \frac{C_{\mathfrak{s},n,p}}{\sqrt{\log(p)}},$$

we obtain

$$\mathbb{P} \left(\max_{k \in T^{*c}} \|X_{T^*}^t X_k\|_2 \geq \frac{C_{col} + (1 + 1.1 \cdot r) C_{\mathfrak{s},n,p}}{\sqrt{\log(p)}} \mid \mathcal{E}_\alpha^* \right) \leq \frac{C+2}{p^\alpha}.$$

Moreover, using Lemma A.1, we obtain

$$\mathbb{P} \left(\max_{k \in T^{*c}} \|X_{T^*}^t X_k\|_2 \geq \frac{C_{col} + (1 + 1.1 \cdot r) C_{\mathfrak{s},n,p}}{\sqrt{\log(p)}} \right) \leq \frac{C+3}{p^\alpha}.$$

C.3. The last two inequalities. The proof of (3.16) is standard and, under Assumption 2.7, the proof of (3.17) can be proved using the ideas of [9, Section 3.3]. We give the proofs for the sake of completeness.

C.3.1. Control of $\|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t z\|_\infty$. For any $j \in T^{*c}$, we have

$$\begin{aligned} \mathbb{P} (X_j^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t z \geq u) &\leq \frac{1}{2} \exp \left(-\frac{u^2}{2\sigma^2 \|X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t X_j\|_2^2} \right) \\ &\leq \frac{1}{2} \exp \left(-\frac{u^2}{2\sigma^2 \frac{1+r^*}{(1-r^*)^2} \frac{(C_{col} + (1+1.1 \cdot r) C_{\mathfrak{s},n,p})^2}{\log(p)}} \right) + \frac{C+219+3}{p^\alpha} \end{aligned}$$

Taking u such that

$$\frac{1}{2} \exp \left(-\frac{u^2}{2\sigma^2 \frac{1+r^*}{(1-r^*)^2} \frac{(C_{col} + (1+1.1 \cdot r) C_{\mathfrak{s},n,p})^2}{\log(p)}} \right) = \frac{1}{p^\alpha}$$

i.e.

$$u = \sqrt{(\alpha \log(p) - \log(2)) 2\sigma^2 \frac{1+r^*}{(1-r^*)^2} \frac{(C_{col} + (1+1.1 \cdot r) C_{\mathfrak{s},n,p})^2}{\log(p)}}.$$

Using the union bound, we finally obtain

$$\begin{aligned} \mathbb{P} \left(\|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t z\|_\infty \geq \sqrt{(\alpha \log(p) - \log(2)) 2\sigma^2 \frac{1+r^*}{(1-r^*)^2} \frac{(C_{col} + (1+1.1 \cdot r) C_{\mathfrak{s},n,p})^2}{\log(p)}} \right) \\ \leq \frac{C+223}{p^{\alpha-1}}. \end{aligned}$$

C.3.2. Control of $\|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} \text{sgn}(\beta_{T^*}^*)\|_\infty$. Hoeffding's inequality gives

$$\begin{aligned} \mathbb{P} (X_j^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} \text{sgn}(\beta_{T^*}^*) \geq u) &\leq \frac{1}{2} \exp \left(-\frac{u^2}{2 \|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t X_j\|_2^2} \right) \\ &\leq \frac{1}{2} \exp \left(-\frac{u^2}{2 \frac{(C_{col} + (1+1.1 \cdot r) C_{\mathfrak{s},n,p})^2}{\log(p) (1-r^*)^2}} \right) + \frac{C+219+3}{p^\alpha}. \end{aligned}$$

Choosing

$$u = \sqrt{(\alpha \log(p) - \log(2)) \frac{(C_{col} + (1 + 1.1 \cdot r) C_{s,n,p})^2}{\log(p) (1 - r^*)^2}}.$$

and applying the union bound, we obtain

$$\mathbb{P} \left(X_j^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} \text{sgn}(\beta_{T^*}^*) \geq \sqrt{(\alpha \log(p) - \log(2)) \frac{(C_{col} + (1 + 1.1 \cdot r) C_{s,n,p})^2}{\log(p) (1 - r^*)^2}} \right) \leq \frac{C + 223}{p^{\alpha-1}}.$$

C.3.3. *Summing up.* Using Assumption 2.6, we obtain that

$$\begin{aligned} & \|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} X_{T^*}^t z\|_\infty + \lambda \|X_{T^*}^t X_{T^*} (X_{T^*}^t X_{T^*})^{-1} \text{sgn}(\beta_{T^*}^*)\|_\infty \\ & \leq \sigma \sqrt{1 + 1.1 \cdot r} (1.1 + 0.11 \cdot r) + \frac{1}{2} \lambda \end{aligned}$$

as announced.

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